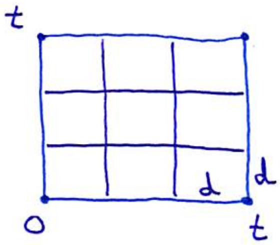
E5.5.3 Let N_A be the number of defects found by inspector A and N_B the number found by inspector B. We have

$$
\begin{aligned}
&\Pr(N_A \geq 1 \text{ AND } N_B \geq 1) \\
&= \Pr(N_A \geq 1)\Pr(N_B \geq 1) \\
&= [1 - \Pr(N_A = 0)][1 - \Pr(N_B = 0)] \\
&= \left[1 - \frac{(\frac{1}{2})^0}{0!} e^{-1/2}\right]^2 \\
&= \left[1 - e^{-\frac{1}{2}}\right]^2 .
\end{aligned}
$$

P5.5.3



box side length $= d = t/n$

number of boxes $= n^2$

Let R denote the number of reactions. Then R takes values in $\{0, \ldots, n^2\}$. The distribution of R is binomial:

$$\Pr(R = k) = \binom{n^2}{k}\left[\sum_{i=2}^{\infty} \frac{(\lambda d^2)^i}{i!} e^{-\lambda d^2}\right]^k \left[\sum_{j=0}^{1} \frac{(\lambda d^2)^j}{j!} e^{-\lambda d^2}\right]^{n^2-k}.$$

Let's try to apply the law of rare events.

① (# successes expected) = (# trials)(Prob. of success)

(A) # trials $= n^2 = t^2/d^2$

(B) Prob. of success $= e^{-\lambda d^2} \sum_{i=2}^{\infty} (\lambda d^2)^i / i!$

$$= \left[1 - \lambda d^2 + \frac{\lambda^2 d^4}{2} - \cdots\right]\left[\frac{\lambda^2 d^4}{2} + \cdots\right].$$

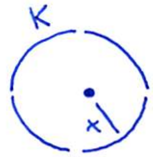
To leading order in d, Prob. of success $\approx \frac{\lambda^2 d^4}{2}$.

② So (# trials)(Prob. of success) $\approx \left(t^2/d^2\right)\left(\frac{\lambda^2 d^4}{2}\right) = \frac{\lambda^2 [dt]^2}{2}$

and this converges to $\frac{(\lambda\mu)^2}{2}$.

By the law of rare events, the distribution of R is Poisson with mean $\frac{(\lambda\mu)^2}{2}$ in the $t \to \infty$; $d \to 0$; $td \to \mu > 0$ limit.

P5.5.5 Let $F_D(x)$ denote the CDF for the distance between a particle and its nearest neighbor.



We have
$$\begin{aligned} F_D(x) &= \Pr(D \le x) \\ &= 1 - \Pr(D > x) \\ &= 1 - \Pr(\#\text{ particles in } K = 0) \\ &= 1 - e^{-\nu\pi x^2}. \end{aligned}$$

To compute the mean distance, we use the tail probability integral
$$\begin{aligned} &\int_0^\infty 1 - F_D(x)\,dx \\ &= \int_0^\infty e^{-\nu\pi x^2}\,dx \qquad \text{substitution: } z^2/2 = \nu\pi x^2 \text{ or } z = (\sqrt{2\nu\pi})x \\ &= \frac{1}{\sqrt{2\pi\nu}}\int_0^\infty e^{-z^2/2}\,dz \\ &= \frac{1}{\sqrt{\nu}}\left[\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-z^2/2}\,dz\right] = \frac{1}{2\sqrt{\nu}}. \end{aligned}$$

P5.5.6 Arguing as in Problem 5.5.5,
$$\begin{aligned} F_R(x) &= \Pr(R \le x) \\ &= 1 - e^{-\lambda[\frac{4}{3}\pi x^3]} \end{aligned}$$

The PDF $f_R(x)$ is given by $f_R(x) = 4\lambda\pi x^2 e^{-\lambda[\frac{4}{3}\pi x^3]}$.

E8.1.3 (a) Let $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. The chain rule implies $\phi'(x) = -x\phi(x)$.

(b)
$$\begin{aligned} \partial^2 p/\partial x^2 &= \frac{1}{t^{3/2}}\phi\left(\frac{y-x}{\sqrt{t}}\right)\left[\frac{(y-x)^2}{t} - 1\right] \\ \partial p/\partial t &= \left(\frac{1}{2t^{3/2}}\right)\phi\left(\frac{y-x}{\sqrt{t}}\right)\left[\frac{(y-x)^2}{t} - 1\right] \end{aligned}$$

Thus $\partial p/\partial t = \frac{1}{2}\partial^2 p/\partial x^2$.

E8.1.4(a) A simple observation: $-B(t)$ is a standard Brownian motion as well.

Thus
$$\begin{aligned} &E[B(u)B(u+v)B(u+v+w)] \\ &= E[(-B(u))(-B(u+v))(-B(u+v+w))] \\ &= -E[B(u)B(u+v)B(u+v+w)], \end{aligned}$$

forcing $E[B(u)B(u+v)B(u+v+w)] = 0$.

E8.1.5(a) Computing the covariance for s, t, we have
$$\begin{aligned} \mathrm{Cov}[U(s)U(t)] &= E[U(s)U(t)] - E[U(s)]E[U(t)] \\ &= e^{-(s+t)}E[B(e^{2s})B(e^{2t})] \\ &= e^{-(s+t)}\min\{e^{2s}, e^{2t}\} = e^{-|s-t|}. \end{aligned}$$