Exam 1: Math 4320 Fall 2017
Professor William Ott

(1) (Points: 20 total, 5 each) Determine if each of the following is true or false.

(a) For any two events A and B, we have $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

(b) For any random variables X and Y, we have $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$.

(c) For any random variable X and any real number a, we have $\text{Var}[aX] = a^2 \text{Var}[X]$.

(d) If X is a discrete random variable that can take finitely many values, then $E[X|X] = X$.

(2) (Points: 20 total, 5 each) Suppose that a Markov chain $X_0, X_1, X_2, X_3, \ldots$ on states $0, 1, 2$ has the transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 0.15 & 0.25 & 0.6 \\ 0.05 & 0.75 & 0.2 \\ 0.35 & 0.55 & 0.1 \end{pmatrix}.$$

Write expressions for the following probabilities in terms of the entries of P. You do not need to simplify your answers - no numerical computations are needed.

(a) $\Pr(X_1 = 2|X_0 = 0)$

(b) $\Pr(X_2 = 2|X_0 = 0)$

(c) $\Pr(X_1 = 2)$ [Assuming the initial distribution is $\Pr(X_0 = 0) = 0.4$ and $\Pr(X_0 = 1) = 0.6$.]

(d) $\Pr(X_0 = 0|X_1 = 2)$ [Assume the same initial distribution.]

(3) (Points: 10) Suppose that a Markov chain $X_0, X_1, X_2, X_3, \ldots$ on states $0, 1, 2$ has the transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $T = \min\{n \geqslant 0 : X_n = 2\}$ denote the absorption time. Write down a system of first-step equations that we would use to solve for $E[T|X_0 = 1]$. You do not need to solve the equations.

(4) (Points: 10) Show that the n-step transition probabilities of a Markov chain satisfy

$$P_{ij}^{(n)} = \sum_{k=0}^{\infty} P_{ik} P_{kj}^{(n-1)}.$$

(5) (Points: 10) Suppose the random variables X and Y are independent and suppose that both are exponentially distributed with parameter $\lambda > 0$. Recall that the probability density function for the exponential distribution is given by

$$f(t) = \begin{cases} 0, & \text{if } t < 0; \\ \lambda e^{-\lambda t}, & \text{if } t \geqslant 0. \end{cases}$$

Find the probability density function for $Z = X + Y$. Hint: perform a convolution.

(6) (Points: 10) Suppose X is a binomial random variable with parameters p (probability of success) and N (number of trials). Suppose N has a Poisson distribution with parameter λ. Write down a formula for $\Pr(X = k)$. You do not need to simplify your formula.

(7) (Points: 10) Suppose we roll a 6-sided die (with faces labeled 1, 2, 3, 4, 5, and 6) repeatedly. Let X_n denote the largest value that appears in the first n rolls. Then X_n is a Markov chain. Write down the transition probability matrix for this Markov chain.