Math 4320 Fall 2017 -- Exam 1 Solutions

1 (a) FALSE (b) FALSE (c) TRUE (d) TRUE

2 (a) $P_{02} = 0.6$

(b) $P_{00}P_{02} + P_{01}P_{12} + P_{02}P_{22} = (0.15)(0.6) + (0.25)(0.2) + (0.6)(0.1)$

(c) $(0.4)(P_{02}) + (0.6)(P_{12}) = (0.4)(0.6) + (0.6)(0.2)$

(d) $$\Pr(X_0=0 \mid X_1=2) = \frac{\Pr(X_0=0 \text{ AND } X_1=2)}{\Pr(X_1=2)}$$

[redacted] Bayes $$= \frac{\Pr(X_1=2 \mid X_0=0)\Pr(X_0=0)}{\Pr(X_1=2)} = \frac{(0.6)(0.4)}{(0.4)(0.6)+(0.6)(0.2)}$$

3 Let $v_1 = E[T \mid X_0=1]$ and $v_0 = E[T \mid X_0=0]$.

The first-step system: $$\begin{cases} v_0 = 1 + (0.2)v_0 + (0.3)v_1 \\ v_1 = 1 + (0.1)v_0 + (0.8)v_1 \end{cases}$$

4 $$\begin{aligned} P_{ij}^{(n)} &= \Pr(X_n=j \mid X_0=i) \\ &= \sum_{k=0}^{\infty} \Pr(X_n=j \text{ AND } X_1=k \mid X_0=i) \\ &= \sum_{k=0}^{\infty} \Pr(X_n=j \mid X_1=k, X_0=i)\Pr(X_1=k \mid X_0=i) \\ &= \sum_{k=0}^{\infty} \Pr(X_n=j \mid X_1=k)\Pr(X_1=k \mid X_0=i) \\ &= \sum_{k=0}^{\infty} P_{ik}P_{kj}^{(n-1)} \quad \square \end{aligned}$$

5 Let h denote the PDF for $X+Y$. We have
$$\begin{aligned} h(s) &= \int_0^s [\lambda e^{-\lambda(s-t)}][\lambda e^{-\lambda t}]\,dt \\ &= \lambda^2 \int_0^s e^{-\lambda s}\,dt = \lambda^2 e^{-\lambda s}\int_0^s dt = \boxed{\lambda^2 s e^{-\lambda s}} \quad \square \end{aligned}$$

6 $$\begin{aligned} \Pr(X=k) &= \sum_{n=k}^{\infty} \Pr(X=k \mid N=n)\Pr(N=n) \quad \text{[law of total probability]} \\ &= \sum_{n=k}^{\infty} \left[\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k}\right]\left[\lambda^n/n!\, e^{-\lambda}\right] \end{aligned}$$

7

Markov matrix:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/2 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$