Exam 2: Math 4320 Fall 2017
Professor William Ott

(1) (Points: 20 total, 5 each) Determine if each of the following is true or false.

(a) Let $\{X(t) : t \geqslant 0\}$ be a Poisson process with rate parameter $\lambda > 0$. For any two times s and t satisfying $0 < s < t$, the random variables $X(s)$ and $X(t)$ are independent.

(b) Consider the random walk on the integers wherein the probability of jumping to the right is p and the probability of jumping to the left is $q = 1 - p$. If $p \neq \frac{1}{2}$, then state 0 is transient.

(c) The transition probability matrix

$$\mathsf{P} = \begin{pmatrix} 0.2 & 0.3 & 0.4 & 0.1 \\ 0 & 0.5 & 0.25 & 0.25 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is regular.

(d) Let $\{X_n\}$ be a branching process that is generated by the offspring random variable ξ. If $E[\xi] = 1$, then the extinction probability u_∞ satisfies $u_\infty = 1$.

(2) (Points: 10) We have two urns that, between them, contain a total of 4 balls. At each step, one of the 4 balls is chosen at random and then moved from the urn it is in to the other urn. Let X_n denote the number of balls in the first urn after the n^{th} step. Write down the transition matrix for this Markov chain.

(3) (Points: 15 total, 5 each) Suppose that

$$\mathsf{P} = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

is a regular transition probability matrix for a Markov chain on states $0, 1, 2$. Let $\pi = (\pi_0, \pi_1, \pi_2)$ denote the limiting distribution. Express the following in terms of the entries of π and P.

(a) In the long run, what fraction of time does the Markov chain spend in state 2?

(b) $\lim_{n\to\infty} \Pr\{X_n = 1 | X_{n+1} = 0\}$. Hint: Bayes formula.

(c) In the long run, what fraction of transitions move from state 1 to state 2?

(4) (Points: 10) Suppose $\xi_1, \xi_2, \ldots$ represent successive bids (independent, identically distributed bids) on a certain asset that is on sale. Let $X_n = \max\{\xi_1, \xi_2, \ldots, \xi_n\}$ be the maximum that has been bid through stage n. Suppose that the accepted bid is the first bid that meets or exceeds a prescribed level M. The time of sale is the random variable $T = \min\{n \geqslant 1 : X_n \geqslant M\}$. Write down a first-step equation for $\mu = E[T]$ in terms of $\Pr\{\xi_1 < M\}$.

(5) (Points: 10, 5 each) Let $\{X(t) : t \geqslant 0\}$ be a Poisson process of rate $\lambda > 0$, where t is measured in hours.

(a) What is the probability that 5 events occur within the first 3 hours and 8 events occur within the first 4 hours?

(b) Assuming $X(t) = 1$, what is the conditional probability that the first event occurs within the time interval $(0, \frac{t}{4}]$?

(6) (Points: 10) Shocks occur to a system according to a Poisson process of rate $\lambda > 0$. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time $t > 0$?

(7) (Points: 10) Consider the random walk $\{X_n\}$ on the integers, where the probability of jumping to the right is $\frac{1}{2}$ and the probability of jumping to the left is also $\frac{1}{2}$. Prove that state 0 is recurrent. You may find the Stirling formula useful:

$$n! \sim n^n e^{-n} (2\pi n)^{1/2}$$

as $n \to \infty$.