Math 4320 Fall 2017 : Exam 2 Solutions

1 (a) False (b) True (c) False (d) True

2

$$\begin{array}{c|ccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1/4 & 0 & 3/4 & 0 & 0 \\ 2 & 0 & 1/2 & 0 & \frac{1}{2} & 0 \\ 3 & 0 & 0 & 3/4 & 0 & \frac{1}{4} \\ 4 & 0 & 0 & 0 & 1 & 0 \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array}$$

3 (a) π_2 (b) $\displaystyle\lim_{n\to\infty} \frac{Pr\{X_n=1, X_{n+1}=0\}}{Pr\{X_{n+1}=0\}} = \frac{\pi_1 a_{10}}{\pi_0}$

(c) $\pi_1 a_{12}$

4

$$\mu = 1 + \mu \cdot Pr\{\xi_1 < M\} + 0 \cdot Pr\{\xi_1 \geq M\}$$

so $\mu = 1 + \mu \cdot Pr\{\xi_1 < M\}$

5 (a) $Pr\{X(3)=5, X(4)=8\} = Pr\{X(3)=5, X(4)-X(3)=3\}$

$$= \left[\frac{(3\lambda)^5}{5!} e^{-3\lambda}\right]\left[\frac{(\lambda)^3}{3!} e^{-\lambda}\right]$$

(b) Let T_1 be the time the first event occurs.

$Pr\{T_1 \in (0, t/4] \mid X(t)=1\}$

$= Pr\{T_1 \in (0, t/4] \text{ AND } X(t)=1\} \,/\, Pr\{X(t)=1\}$

$$= \left[\frac{(t/4\,\lambda)^1}{1!} e^{-t/4\,\lambda}\right]\left[\frac{(3t/4\cdot\lambda)^0}{0!} e^{-3t/4\cdot\lambda}\right] \Big/ \left[\frac{(\lambda t)^1}{1!} e^{-\lambda t}\right]$$

$= 1/4$.

6 $Pr\{\text{system alive at time } t\} = \displaystyle\sum_{k=0}^{\infty} \alpha^k \left[\frac{(\lambda t)^k}{k!} e^{-\lambda t}\right]$

$$= \left[\sum_{k=0}^{\infty} \frac{(\alpha\lambda t)^k}{k!}\right] \cdot e^{-\lambda t}$$

$$= e^{\alpha\lambda t} e^{-\lambda t} = e^{\lambda t[\alpha - 1]}$$

7 Let $P_{00}^{(n)}$ denote the probability of being in state 0 at time n, given $X_0 = 0$. We will show $\sum P_{00}^{(n)}$ diverges. Since returns to 0 are only possible at even times, the series becomes

$$\sum_{n=1}^{\infty} \binom{2n}{n} \left(\tfrac{1}{2}\right)^{2n}$$

$$\sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi}\cdot\sqrt{n}} \quad \text{(using Stirling)}.$$

This series diverges. □