Exam 3: Math 4320 Fall 2017
Professor William Ott

(1) (Points: 20 total, 5 each) Determine if each of the following is true or false.

- **(a)** If $\{B(t) : t \geqslant 0\}$ is standard Brownian motion and c is any positive constant, then $\{cB(t/c) : t \geqslant 0\}$ is standard Brownian motion as well.
- **(b)** Let $\{B(t) : t \geqslant 0\}$ denote standard Brownian motion. For any times $0 \leqslant t_1 < t_2 < t_3 < t_4$, we have $\mathrm{Cov}[B(t_4) - B(t_3), B(t_2) - B(t_1)] = 0$.
- **(c)** Let $\{X(t) : t \geqslant 0\}$ be a Poisson process with rate parameter $\lambda > 0$. Then $\Pr\{X(5) = 7/2\} = 0$.
- **(d)** Let $\{X(t) : t \geqslant 0\}$ be a Poisson process with rate parameter $\lambda > 0$. Assuming it is known that $X(T) = 1$ for some positive time T, then W_1, the time of the first event, has conditional PDF $f(t) = \lambda e^{-\lambda t}$ $(t \geqslant 0)$.

(2) (Points: 30 total, 5 each) Let $\{X(t) : t \geqslant 0\}$ be a Poisson process with rate parameter λ.

- **(a)** Find $\Pr\{X(7) = 3\}$.
- **(b)** Find $\Pr\{X(7) = 3 | X(2) = 1\}$.
- **(c)** Find $\Pr\{X(2) = 1 \text{ and } X(7) = 3\}$.
- **(d)** Let W_1 denote the time of the first event. Find $\Pr\{a \leqslant W_1 \leqslant b\}$, where $0 \leqslant a < b$ are two times.
- **(e)** Find $\Pr\{W_1 \leqslant 10 | X(50) = 1\}$.
- **(f)** Let W_1 and W_2 denote the times of the first and second events, respectively. Find $\Pr\{W_1 \leqslant 1 \text{ and } W_2 \leqslant 1 | X(3) = 2\}$.

(3) (Points: 10) Consider a three-dimensional spatial Poisson process (points in space) with intensity parameter ν. Let D denote the distance between a given particle and its nearest neighbor. For $r > 0$, find $\Pr\{D \leqslant r\}$. You may use the fact that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Hint: For any event A, $\Pr(A) = 1 - \Pr(A^C)$, where A^C denotes the complement of A.

(4) (Points: 10) Professor Ott walks along the x-axis in search of his favorite sports bar, which is located at $x = 4$. He starts walking from $x = 0$ at time zero. Because he has eaten a substantial number of Starburst candy chews, his motion is described by standard Brownian motion $\{B(t) : t \geqslant 0\}$. Use the reflection principle to write down an integral for the probability that Professor Ott reaches the sports bar during the first 9 time units of his walk. You do not need to evaluate the integral.

(5) (Points: 10, 5 each) Let $\{X(t) : t \geqslant 0\}$ be a Poisson process with rate parameter λ and let W_1 denote the time of the first event. We will find the conditional PDF for W_1 assuming $X(T) = 1$.

- **(a)** For $0 \leqslant s \leqslant T$, let $F(s) = \Pr\{W_1 \leqslant s | X(T) = 1\}$. Find $F(s)$.
- **(b)** Differentiate $F(s)$. (This gives the conditional PDF for W_1 assuming $X(T) = 1$.

(6) (Points: 10) Shocks occur to a system according to a Poisson process of rate $\lambda > 0$. Suppose that the system survives each shock with probability α, independently of other shocks, so that its probability of surviving k shocks is α^k. What is the probability that the system is surviving at time $t > 0$? Hint: law of total probability.