|1.4.5h| Let x and y be integers. Prove that if x is even and y is odd, then $x+y$ is odd.

PROOF. Let x and y be integers. Suppose that x is even and y is odd. Since x is even, $x = 2k$ for some integer k. Since y is odd, $y = 2l+1$ for some integer l. Then

$$x+y = 2k + (2l+1) = (2k+2l)+1 = 2(k+l)+1.$$

Since $k+l$ is an integer, we conclude that $x+y$ is odd. ■

|1.4.6d| Let a and b be real numbers. Prove that $|a+b| \leq |a| + |b|$.

PROOF. Since the absolute value function is defined piecewise, we consider four cases.

Case 1: Assume $a \geq 0$ and $b \geq 0$. Then $a+b \geq 0$, so

$$|a+b| = a+b = |a| + |b|.$$

Case 2: Assume $a<0$ and $b<0$. Then $a+b<0$, so

$$|a+b| = -(a+b) = -a+(-b) = |a|+|b|.$$

Case 3: Assume $a \geq 0$ and $b<0$. There are two subcases to consider.

Subcase A: If $a+b \geq 0$, then

$$|a+b| = a+b < a+|b| = |a|+|b|.$$

Subcase B: If $a+b<0$, then

$$\begin{aligned} |a+b| &= -(a+b) = -a+(-b) \\ &= -a+|b| \leq |a|+|b|. \square \end{aligned}$$

Case 4: If $a<0$ and $b \geq 0$, argue exactly as in Case 3.

|1.4.7k| Let a, b, c, d be integers. Prove that if $a \mid b$ and $c \mid d$, then $ac \mid bd$.

Proof. Assume $a \mid b$ and $c \mid d$. Then there exist integers m and n such that $b = am$ and $d = cn$. We have

$$\begin{aligned} bd &= (am)(cn) \\ &= (ac)(mn). \end{aligned}$$

Since mn is an integer, we conclude that $ac \mid bd$. $\square$

(1.4.8 b) Let n be a natural number. Prove that n^2+n+3 is odd.

PROOF. Let n be a natural number. We have

$$n^2+n+3 = n(n+1)+3.$$

By 1.4.7d, $n(n+1)$ is even. Therefore, $[n(n+1)]+3$ is odd by 1.4.5h. We conclude that n^2+n+3 is odd.

(1.4.9 a) If x and y are nonnegative real numbers, then

$$(*) \qquad \frac{x+y}{2} \geq \sqrt{xy}.$$

SCRATCH

Let us transform (*).

$$\begin{aligned} & \frac{x+y}{2} \geq \sqrt{xy} \\ \Leftrightarrow \quad & x+y \geq 2\sqrt{xy} \\ \Leftrightarrow \quad & x+y-2\sqrt{xy} \geq 0 \\ \Leftrightarrow \quad & (\sqrt{x}-\sqrt{y})^2 \geq 0. \end{aligned}$$

PROOF. Suppose x and y are nonnegative real numbers. Then

$$\begin{aligned} & (\sqrt{x}-\sqrt{y})^2 \geq 0 \\ \Rightarrow \quad & x+y-2\sqrt{xy} \geq 0 \\ \Rightarrow \quad & x+y \geq 2\sqrt{xy} \\ \Rightarrow \quad & \frac{x+y}{2} \geq \sqrt{xy}. \quad \square \end{aligned}$$