Math 3325 Fall 2019 

Assignment 5 Solutions

**2.2.15a** Prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof: We can use a sequence of if and only if (iff) statements here. We have

$$\begin{aligned}
& (p,q) \in A \times (B \cap C) \\
\text{iff } & p \in A \text{ and } q \in B \cap C \\
\text{iff } & p \in A \text{ and } (q \in B \text{ and } q \in C) \\
\text{iff } & (p \in A \text{ and } q \in B) \text{ and } (p \in A \text{ and } q \in C) \\
\text{iff } & (p,q) \in A \times B \text{ and } (p,q) \in A \times C \\
\text{iff } & (p,q) \in (A \times B) \cap (A \times C). \quad \square
\end{aligned}$$

**2.3.1dh** For $n \in \mathbb{N}$, let $B_n = \mathbb{N} - \{1, \ldots, n\} = \{n+1, n+2, \ldots\}$.

Define $\hat{B} = \{B_n : n \in \mathbb{N}\}$.

- $\bigcup_{n \in \mathbb{N}} B_n = \mathbb{N} - \{1\} = \{k \in \mathbb{N} : k \geq 2\} = \{2, 3, 4, \ldots\}$.
- $\bigcap_{n \in \mathbb{N}} B_n = \emptyset$

(h) For $r \in (0, \infty)$, define $A_r = [-\pi, r)$. Let $\hat{A} = \{A_r : r \in (0, \infty)\}$.

- $\bigcup_{r \in (0,\infty)} A_r = [-\pi, \infty)$
- $\bigcap_{r \in (0,\infty)} A_r = [-\pi, 0]$

**2.3.12** For example, let $A_n = (0, \frac{1}{n})$. Then

- $A_n \subset (0,1)$ for every $n \in \mathbb{N}$;
- if $m, n \in \mathbb{N}$, and we assume that $m \geq n$, then $A_m \cap A_n = A_m$, so $A_m \cap A_n \neq \emptyset$;
- $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

**2.3.15 cf** Let $\hat{A} = \{A_i : i \in \mathbb{N}\}$ be a family of sets and let $k, m \in \mathbb{N}$ satisfy $k \leq m$.

(c) $\bigcup_{i=k}^{m} A_i \subset \bigcup_{i=1}^{\infty} A_i$.

Proof: Let $p \in \bigcup_{i=k}^{m} A_i$. Then there exists $j \in \mathbb{N}$ such that $k \leq j \leq m$, and $p \in A_j$. Therefore $p \in A_i$ for some $i \in \mathbb{N}$, so $p \in \bigcup_{i=1}^{\infty} A_i$. □

(f) $\bigcap_{i=1}^{m} A_i \subset \bigcap_{i=1}^{k} A_i$.

Proof: Let $p \in \bigcap_{i=1}^{m} A_i$. Then $p \in A_i$ for every $i \in \mathbb{N}$ satisfying $1 \leq i \leq m$. But $k \leq m$, so $p \in A_i$ for every $i \in \mathbb{N}$ satisfying $1 \leq i \leq k$, and therefore

$$p \in \bigcap_{i=1}^{k} A_i. \quad \square$$

**2.3.16 a** Suppose $\{A_i : i \in \mathbb{N}\}$ is a decreasing nested family of sets.

Claim: For every $k \in \mathbb{N}$, $\bigcap_{i=1}^{k} A_i = A_k$.

Proof: ($\subset$)
Let $p \in \bigcap_{i=1}^{k} A_i$. Then $p \in A_i$ for every $i \in \mathbb{N}$ satisfying $1 \leq i \leq k$. In particular, $p \in A_k$.

($\supset$) Let $p \in A_k$. Let $i \in \mathbb{N}$ satisfy $1 \leq i \leq k$. By hypothesis, $A_k \subset A_i$, so $p \in A_k$ implies $p \in A_i$. Since $p \in A_i$ for every $i \in \mathbb{N}$ satisfying
$1 \leq i \leq k$,
we conclude $p \in \bigcap_{i=1}^{k} A_i$. □