Math 3325 Fall 2019
Assignment 6 Solutions

2.4.6c $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$. $(*)$

Proof: We use PMI. For the base case, assume $n=1$. We have

$$\sum_{i=1}^{1} 2^i = 2^1 = 2 = 2^{1+1} - 2.$$

Now assume $(*)$ holds for $n=k$. We have

$$\begin{aligned}\sum_{i=1}^{k+1} 2^i &= \sum_{i=1}^{k} 2^i + 2^{k+1} \\ &= (2^{k+1} - 2) + 2^{k+1} \quad \text{(inductive assump.)} \\ &= 2 \cdot 2^{k+1} - 2 \\ &= 2^{k+2} - 2. \ \square\end{aligned}$$

2.4.7b Prove that for every $n \in \mathbb{N}$, $4^n - 1$ is divisible by 3.

Proof: We use PMI. For the base case, set $n=1$. Then $4^1 - 1 = 3$, and certainly $3 \mid 3$. Now assume $3 \mid 4^k - 1$. This means that there exists $q \in \mathbb{Z}$ for which $(4^k - 1) = 3q$. We have

$$\begin{aligned}4^{k+1} - 1 &= 4^{k+1} - 4^k + 4^k - 1 \\ &= 4^k(4-1) + (4^k - 1) \\ &= 4^k(3) + 3q \\ &= 3(4^k + q).\end{aligned}$$

Since $4^k + q \in \mathbb{Z}$, we conclude that $3 \mid (4^{k+1} - 1)$. $\square$

2.4.8c Claim: $(n+1)! > 2^{n+3}$ for $n \geq 5$

Proof: If $n=5$, we have

$$6! = 720 > 2^8 = 256.$$

Now let $k \in \mathbb{N}$ satisfy $k \geq 5$ and assume $(k+1)! > 2^{k+3}$. We have

$$\begin{aligned}(k+2)! &= (k+2)(k+1)! \\ &= (k+2)[(k+1)k(k-1)\cdots 1] \\ &> (k+2)\, 2^{k+3} \\ &> 2 \cdot 2^{k+3} \\ &= 2^{k+4}.\end{aligned}$$

We conclude that $(k+2)! > 2^{k+4}$. $\square$

2.5.2 Let $a_1 = 2$, $a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 1$. Prove that $a_n = 2^n$ for all $n \in \mathbb{N}$.

Proof: The base cases $n=1$, $n=2$ hold by definition. Now let $k \geq 2$ and assume that

$$a_i = 2^i$$

for every i satisfying $1 \leq i \leq k$. We have

$$\begin{aligned} a_{k+1} &= 5a_k - 6a_{k-1} \\ &= 5(2^k) - 6(2^{k-1}) \\ &= 2^{k-1}[5 \cdot 2 - 6] \\ &= 2^{k-1}[4] \\ &= 2^{k+1}. \quad \square \end{aligned}$$

2.5.6d We show that the n^{th} Fibonacci number f_n is given by $f_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, where α and β are the positive and negative solutions, respectively, of $x^2 = x + 1$.

Proof: First we treat the base cases.

$$f_1 = 1 = (\alpha - \beta)/(\alpha - \beta).$$

$$f_2 = 1 = \left(\tfrac{1}{2}(1+\sqrt{5}) + \tfrac{1}{2}(1-\sqrt{5})\right) = \alpha + \beta = \frac{\alpha^2 - \beta^2}{\alpha - \beta}.$$

Now assume $k \geq 2$ and

$$f_i = (\alpha^i - \beta^i)/(\alpha - \beta)$$

for every $i \in \mathbb{N}$ satisfying $1 \leq i \leq k$. We have

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &= \frac{\alpha^k - \beta^k}{\alpha - \beta} + \frac{\alpha^{k-1} - \beta^{k-1}}{\alpha - \beta} \\ &= \frac{\alpha^{k-1}(1+\alpha) + \beta^{k-1}(-\beta - 1)}{\alpha - \beta} \\ &= \frac{\alpha^{k-1}(\alpha^2) + \beta^{k-1}(-\beta^2)}{\alpha - \beta} \\ &= \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta}. \quad \square \end{aligned}$$