Math 3325 Fall 2019
Assignment 8 Solutions

(4.3.3.d) Define $a : \mathbb{N} \to \mathbb{N}$ by $a_n = 2^n$. First, a is NOT onto. To see this, notice that $3 \notin \text{Range}(a)$, since a_n is even for all $n \in \mathbb{N}$. Second, a IS one-to-one. To see this, suppose $n, m \in \mathbb{N}$ and

$$2^n = 2^m.$$

Then $\ln(2^n) = \ln(2^m)$, so $n \cdot \ln(2) = m \cdot \ln(2)$, and therefore $n = m$. $\square$

(4.3.5) Suppose $f : A \to B$ and $g : B \to C$ are onto. We show that $g \circ f : A \to C$ is onto. Let $w \in C$. Since g is onto, $\exists\, b \in B$ such that $g(b) = w$. Since f is onto, $\exists\, a \in A$ such that $f(a) = b$. Then $(g \circ f)(a) = g(f(a)) = g(b) = w$. We conclude that Range $(g \circ f) = C$. $\square$

(4.4.2.e) Define $f : 12\mathbb{N} \to 20\mathbb{N}$ by $f(x) = \frac{5}{3}x$. We first show that f is onto. Let $p \in 20\mathbb{N}$. Then $p = 20k$ for some $k \in \mathbb{N}$. We have that $12k \in 12\mathbb{N}$, and $f(12k) = 20k = p$. Thus f is ONTO. To see that f is one-to-one, let $x_1, x_2 \in 12\mathbb{N}$ and assume $f(x_1) = f(x_2)$. Then $\frac{5}{3}x_1 = \frac{5}{3}x_2$, so $x_1 = x_2$. $\square$

(4.5.10.a) Let $f : A \to B$. Suppose $E \subset B$. We show that $f(f^{-1}(E)) \subset E$. Let $y \in f(f^{-1}(E))$. Then $\exists\, x \in f^{-1}(E)$ such that $y = f(x)$. But $x \in f^{-1}(E)$, so $f(x) \in E$. We conclude that $y \in E$.

(4.5.12) Let $f : A \to B$. We show that if f is one-to-one, then $f(X) \cap f(Y) = f(X \cap Y)$ for all $X, Y \subset A$.

Proof: Assume $X, Y \subset A$ and assume f is one-to-one. We have $f(X \cap Y) \subset f(X) \cap f(Y)$ by Theorem 4.5.1. We therefore need $f(X) \cap f(Y) \subset f(X \cap Y)$. Let $p \in f(X) \cap f(Y)$. Then $\exists\, x, z \in A$ such that $x \in X$, $z \in Y$, and $f(x) = f(z) = p$. But f is one-to-one, so $x = z$. We conclude that

We conclude that $z \in X \cap Y$ and so $p \in f(X \cap Y)$. This completes the proof. $\square$

Converse? The converse is true.

Claim: Let $f: A \to B$. Prove that if $f(X) \cap f(Y) = f(X \cap Y)$ for all $X, Y \subset A$, then f is one-to-one.

Proof: Let $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$. Let us use y to denote $f(a_1)$. Using the hypothesis with $X = \{a_1\}$, $Y = \{a_2\}$, we have

$$f(X) \cap f(Y) = \{y\} \cap \{y\} = \{y\},$$

so $f(\{a_1\} \cap \{a_2\}) = \{y\}$.

We must therefore have $a_1 = a_2$, since otherwise $\{a_1\} \cap \{a_2\} = \phi$, and $f(\phi) = \phi$. $\square$