Math 3325 Fall 2019
Assignment 9 Solutions

**5.1.4** Suppose $h: A \to C$ and $g: B \to D$ are one-to-one correspondences. We prove that $f: A \times B \to C \times D$ defined by $f(a,b) = (h(a), g(b))$ is a one-to-one correspondence.

Proof. Let $(z_1, z_2) \in C \times D$. Since $h$ is onto, there exists $x_1 \in A$ such that $h(x_1) = z_1$. Since $g$ is onto, $\exists\ x_2 \in B$ such that $g(x_2) = z_2$. Then $(x_1, x_2) \in A \times B$ and $f(x_1, x_2) = (z_1, z_2)$. This shows that $f$ is onto. Now suppose $(p_1, p_2), (q_1, q_2) \in A \times B$ and $f(p_1, p_2) = f(q_1, q_2)$. This implies $h(p_1) = h(q_1)$, and $h$ is one-to-one, so $p_1 = q_1$. Similarly, we have $g(p_2) = g(q_2)$ and $g$ is one-to-one, so $p_2 = q_2$. We conclude that $(p_1, p_2) = (q_1, q_2)$, so $f$ is one-to-one.

**5.2.4 (b)** Let $a \in \mathbb{R}$. Define $f: (0,1) \to (a, \infty)$ by $f(x) = \tan(\frac{\pi}{2}x) + a$. This function is a one-to-one correspondence, so $\mathrm{card}((a,\infty)) = c$.

**[5.2.7 (acdef)]** (a) $c$ (b) not assigned; $\mathrm{card}((5,\infty)) = c$.
(c) $\aleph_0$
(d) $\aleph_0$
(e) $c$
(f) $c$

**5.4.3** Let $A, B, C$ be sets and assume $\mathrm{card}(A) \leq \mathrm{card}(B)$ and $\mathrm{card}(B) = \mathrm{card}(C)$. We show that $\mathrm{card}(A) \leq \mathrm{card}(C)$.

Proof. Since $\mathrm{card}(A) \leq \mathrm{card}(B)$, there exists a one-to-one function $f: A \to B$.
Since $\mathrm{card}(B) = \mathrm{card}(C)$, $\exists$ a one-to-one, onto function $g: B \to C$.

Since $f$ and $g$ are both one-to-one, so is $g \circ f : A \to C$. Therefore $\text{card}(A) \le \text{card}(C)$.

$\boxed{5.4.15(c)}$ Suppose $A \subset \mathbb{R}$ and $\exists$ an open interval $(a, b)$ such that $(a, b) \subset A$. We claim that $\text{card}(A) = c$.

First, the function $i : A \to \mathbb{R}$ defined by $i(x) = x$ is one-to-one, so $\text{card}(A) \le \text{card}(\mathbb{R})$.

Second, recall that $f : (0,1) \to \mathbb{R}$ defined by $f(x) = \tan(\pi x - \pi/2)$ is a one-to-one correspondence. Define $h : (0,1) \to (a,b)$ by

$$h(x) = a + (b-a)x.$$

Then $h$ is a one-to-one correspondence between $(0,1)$ and $(a,b)$. The function $h \circ f^{-1} : \mathbb{R} \to A$ is one-to-one, so $\text{card}(\mathbb{R}) \le \text{card}(A)$.

We have shown $\text{card}(A) \le \text{card}(\mathbb{R})$ and $\text{card}(\mathbb{R}) \le \text{card}(A)$, so Cantor-Schröder-Bernstein implies $\text{card}(A) = \text{card}(\mathbb{R}) = c$.