Exam 2: Math 3325 Fall 2019
Professor William Ott

Exercise 1. (10, 5 each) (Definitions) Let A be a nonempty set.

(a) Give the definition of an ***equivalence relation*** R on A.

(b) Suppose R is an equivalence relation on A. Let $x \in A$. Give the definition of the ***equivalence class*** $[x]$.

Exercise 2. (10) Let A, B, and C be sets. Prove that if $A \subset B \cup C$ and if $A \cap B = \emptyset$, then $A \subset C$.

Exercise 3. (10, 5 each) Find the following sets.

(a) $\bigcup_{n=1}^{\infty} \left[0, 2 - \frac{1}{n}\right)$

(b) $\bigcap_{n=1}^{\infty} [n, \infty)$

Exercise 4. (10) Use the PMI to show that for all natural numbers n, we have

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1.$$

Recall that for any natural number m, $m!$ denotes the product $(m)(m-1)\cdots(2)(1)$.

Exercise 5. (10) Define the sequence $(a_n)_{n=1}^{\infty}$ by $a_1 = 2$, $a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$ for every $n \geqslant 1$. Use the PCI to prove that $a_n = 2^n$ for every $n \in \mathbb{N}$.

Exercise 6. (10) Prove that the set P of prime numbers is an infinite set.

Exercise 7. (10, 5 each) Determine if each of the following statements is true or false. If true, explain your answer. If false, provide a counterexample.

(a) The relation S on the set of integers $\mathbb{Z}$ defined by xSy iff $x \neq y$ is transitive.

(b) Suppose V is a set that contains 10 elements. If a certain equivalence relation on V produces at least one equivalence class with 4 elements, at least one equivalence class with 3 elements, and at least one equivalence class with 2 elements, then the partition of V produced by this equivalence relation consists of exactly 4 sets.

Exercise 8. (10 total, 5 each) Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Define the relation T on $\mathbb{R}^*$ by xTz iff $\frac{1}{|x|} = \frac{1}{|z|}$.

(a) Prove that T is an equivalence relation on $\mathbb{R}^*$.

(b) Find $[5]$.