Math 3325 Fall 2019
Exam 2 Solutions

E1 An equivalence relation R on A is a relation on A that is reflexive, symmetric, and transitive.

(b) $[x] = \{ y \in A : xRy \}$

E2 Suppose $A \subset B \cup C$ and $A \cap B = \emptyset$. We show $A \subset C$. Let $x \in A$. Then $A \subset B \cup C$ implies $x \in B \cup C$, so $x \in B$ or $x \in C$. But $x \in A$, and $A \cap B = \emptyset$, so $x \notin B$. We conclude that $x \in C$.

E3 (a) $[0, 2)$

(b) $\emptyset$

E4 We use PMI. For the base case $n = 1$,

$$1 \cdot 1! = 2! - 1 = 1.$$

Now assume the result for $k \in \mathbb{N}$. We have

$$\begin{aligned} 1 \cdot 1! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= [(k+1)! - 1] + (k+1) \cdot (k+1)! \\ &= (k+2) \cdot (k+1)! - 1 \\ &= (k+2)! - 1 \quad \square \end{aligned}$$

E5 We use PCI. The base cases $n=1$ and $n=2$ hold by definition. Now let $k \in \mathbb{N}$ and assume the result holds for all $1 \le i \le k$. If $k \ge 2$, we have

$$\begin{aligned} a_{k+1} &= 5a_k - 6a_{k-1} \\ &= 5 \cdot 2^k - 6 \cdot 2^{k-1} \\ &= 2^{k-1}[5 \cdot 2 - 6] \\ &= 2^{k-1}[4] \\ &= 2^{k+1}. \quad \square \end{aligned}$$

E6 Suppose the set of primes is finite, say $P = \{ p_1, \ldots, p_L \}$. Define $s = p_1 \cdots p_L + 1$. Then $s \notin P$, so there exists $p_j \in P$ such that $p_j \mid s$. But $p_j \mid (p_1 \cdots p_L)$ also, so

$$p_j \mid [s - (p_1 \cdots p_L)] = 1.$$

Contradiction.

Math 3325 Fall 2019
Exam 2 Solutions

E7 (a) False

For example, $3 S 7$ and $7 S 3$, but 3 is not related to 3.

(b) True

It is possible only to have a 4^{th} equivalence class that contains exactly one element.

E8 (a) (Reflexive)

Let $x \in \mathbb{R}^*$. Then $\frac{1}{|x|} = \frac{1}{|x|}$, so $x T x$.

(Symmetry)

Let $x, y \in \mathbb{R}^*$ and assume $x T y$. Then $\frac{1}{|x|} = \frac{1}{|y|}$, so $\frac{1}{|y|} = \frac{1}{|x|}$ and therefore $y T x$.

(Transitive) Let $x, y, z \in \mathbb{R}^*$ and suppose $x T y$ and $y T z$. Then

$$\frac{1}{|x|} = \frac{1}{|y|} \text{ and } \frac{1}{|y|} = \frac{1}{|z|},$$

so $\frac{1}{|x|} = \frac{1}{|z|}$ and therefore $x T z$.

(b) $[5] = \{-5, 5\}$.