Exam 3: Math 3325 Fall 2019
Professor William Ott

Exercise 1. (10, 5 each) (Definitions)

- **(a)** Give the definition of a ***one-to-one*** function $f : A \to B$.
- **(b)** State what it means for two sets A and B to be ***equivalent***, written $A \approx B$. (Here I am referring to equivalence in the sense of cardinality theory.)

Exercise 2. (15, 3 each) Determine if each of the following statements is true or false.

- **(a)** For every set A, we have $\mathrm{card}(A) \leqslant \mathrm{card}(\mathcal{P}(A))$.
- **(b)** If A is a countably infinite set and $A \subset B$, then B is countably infinite.
- **(c)** Let $f : A \to B$ and $g : B \to C$ be functions. If $g \circ f : A \to C$ is one-to-one, then f is one-to-one.
- **(d)** $(0, 1) \approx \mathbb{R}$
- **(e)** If $h : \mathbb{R} \to \mathbb{R}$ is defined by $h(x) = x^2$, then $h^{-1}([16, 25]) = [4, 5]$.

Exercise 3. (15) Define the relation S on $\{2, 3, 6\}$ by xSy iff $x + y > 7$.

- **(a)** **(3)** Draw the directed graph corresponding to S.
- **(b)** **(4)** Is S reflexive?
- **(c)** **(4)** Is S symmetric?
- **(d)** **(4)** Is S transitive?

For the final three parts, you must either prove your assertion or provide a counterexample.

Exercise 4. (15, 3 each) (Cardinality detection) For each given set A, determine if $\mathrm{card}(A) = \aleph_0$ or $\mathrm{card}(A) = \boldsymbol{c}$.

- **(a)** The set $\mathbb{Z}$ of integers
- **(b)** The set $\mathbb{Q}$ of rational numbers
- **(c)** $\{(p, q) \in \mathbb{R} \times \mathbb{R} : p + q = 1\}$
- **(d)** A is a subset of $\mathbb{R}$ for which there exist real numbers $c < d$ such that $(c, d) \subset A$.
- **(e)** The set of all subsets of $\mathbb{N}$ (the power set of $\mathbb{N}$)

You do not need to prove your assertions.

Exercise 5. (10, 5 each) Define $a : \mathbb{N} \to \mathbb{N}$ by $a_n = 2^n$.

- **(a)** Is a one-to-one?
- **(b)** Is a onto?

You must prove your answer to each question.

Exercise 6. (10) Suppose that $f : A \to B$ and $g : B \to C$ are both onto. Prove that the composition $g \circ f : A \to C$ is onto.

Exercise 7. (10) Let

$$E^+ = \{2k : k \in \mathbb{N}\} = \{2, 4, 6, \ldots\}$$

be the set of positive even integers. Carefully prove that E^+ is a countably infinite set. (In other words, carefully prove that $\mathrm{card}(E^+) = \aleph_0$.)