Math 6320 Fall 2019
Assignment 1 Solutions

$\boxed{\text{Bass 2.2}}$ Set $X = [0,1]$. The collections

$$\hat{A}_1 = \{ \emptyset, [0,1], [0, 3/4), [3/4, 1] \}$$
$$\hat{A}_2 = \{ \emptyset, [0,1], [0, 1/4], (1/4, 1] \}$$

are σ-algebras, but $\hat{A}_1 \cup \hat{A}_2$ is not a σ-algebra, since $[0, 3/4) \cap (1/4, 1] = (1/4, 3/4) \notin \hat{A}_1 \cup \hat{A}_2$.

$\boxed{\text{Bass 2.3}}$ Let us work with $\mathbb{N}$. For each $n \in \mathbb{N}$, let $\hat{C}_n$ be the collection of all subsets of $\{1, \dots, n\}$. Let $\hat{A}_n$ be the σ-algebra {(with underlying space $\mathbb{N}$)} generated by $\hat{C}_n$. We have

$$\hat{A}_n = \{ A \subset \mathbb{N} : A \subset \{1, \dots, n\}\} \cup \{ A \cup \{n+1, n+2, n+3, \dots\} : A \subset \{1, \dots, n\}\}.$$

By construction, $\hat{A}_1 \subset \hat{A}_2 \subset \hat{A}_3 \subset \cdots$. However, $\bigcup_{n=1}^{\infty} \hat{A}_n$ is not a σ-algebra, since $\{2k : k \in \mathbb{N}\} \notin \bigcup_{n=1}^{\infty} \hat{A}_n$, but each element of this set does lie in $\bigcup_{n=1}^{\infty} \hat{A}_n$.

$\boxed{\text{Bass 2.5}}$ The idea here is that preimages behave well.

1. $\emptyset \in \hat{B}$, since $\emptyset = f^{-1}(\emptyset)$ and $\emptyset \in \hat{A}$.
2. Let $B \in \hat{B}$. Then $B = f^{-1}(A)$ for some $A \in \hat{A}$. Since $Y \setminus A \in \hat{A}$, we have $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A) \in \hat{B}$.
3. Let $(B_i)_{i=1}^{\infty}$ be a sequence of sets in $\hat{B}$. For each $i \in \mathbb{N}$, $B_i = f^{-1}(A_i)$ for some $A_i \in \hat{A}$. Then $\bigcup_{i=1}^{\infty} A_i \in \hat{A}$, so $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} f^{-1}(A_i) = f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) \in \hat{B}$.

$\boxed{\text{Bass 2.7}}$ First, $\emptyset \in \hat{A}$, since $\chi_{\emptyset}$ is the zero function, and F contains all constant functions. Second, suppose $A \in \hat{A}$. Then $\chi_A \in F$, so $\chi_{A^c} = 1 - \chi_A \in F$, since F is closed with respect to scalar multiplication and addition, and $1 \in F$. We conclude that $A^c \in \hat{A}$.
It remains to show that $\hat{A}$ is closed with respect to countable unions.

Let (A_i) be a sequence of sets in $\hat{A}$. We first show that for each $n \in \mathbb{N}$, $\bigcup_{i=1}^n A_i \in \hat{A}$. Let $n \in \mathbb{N}$. Let Σ_n be the space of sequences of length n consisting of zeros and ones. We have the representation

$$\chi_{\bigcup_{i=1}^n A_i} = \sum_{\substack{\sigma \in \Sigma_n \\ \sigma_i = 1 \text{ for some} \\ 1 \leq i \leq n}} \prod_{j=1}^n \chi_{E_j},$$

where $E_j = A_j$ if $\sigma_j = 1$ and $E_j = A_j^c$ if $\sigma_j = 0$. By our assumptions on F and since $\hat{A}$ is closed with respect to complements, $\chi_{\bigcup_{i=1}^n A_i} \in F$. But then $\chi_{\bigcup_{i=1}^\infty A_i}$ is the pointwise limit of the sequence $\chi_{\bigcup_{i=1}^n A_i}$, so $\chi_{\bigcup_{i=1}^\infty A_i} \in F$, since F is closed with respect to pointwise limits. We conclude that $\bigcup_{i=1}^\infty A_i \in \hat{A}$.

Bass 2.8 We proceed by way of contradiction. Suppose there exists a countably infinite σ-algebra $\hat{A} = \{A_1, A_2, \ldots\}$. Let Σ denote the set of sequences $(\gamma_i)_{i=1}^\infty$ of zeros and ones. For each $(\gamma_i)_{i=1}^\infty \in \Sigma$, define the set $S_{(\gamma_i)} = \bigcap_{i=1}^\infty E_i$, where $E_i = A_i$ if $i=1$ and $E_i = A_i^c$ if $i = 0$. Note that $S_{(\gamma_i)} \in \hat{A}$ for every $(\gamma_i) \in \Sigma$. One can think of the $S_{(\gamma_i)}$ as "atoms": Every element of $\hat{A}$ may be expressed as a union of such atoms. Let $\hat{S} = \{S_{(\gamma_i)} : (\gamma_i) \in \Sigma\}$. There are two cases to consider.

Case 1: $\hat{S}$ is a finite collection. This would imply that $\hat{A}$ is a finite collection, a contradiction.

Case 2: $\hat{S}$ is infinite. In this case, $\hat{A}$ would be uncountable, a contradiction. $\square$

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Bass 3.2 It remains to show that μ is countably additive. Let $(A_i)_{i=1}^{\infty}$ be a sequence of pairwise-disjoint measurable sets. Define $E_n = \bigcup_{i=n+1}^{\infty} A_i$. Then the sequence (E_n) decreases to ϕ, so

$$\lim_{n\to\infty} \mu(E_n) = 0, \quad \text{or}$$

$$(*) \qquad \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{n} A_i \right) = 0.$$

Since $\mu(X) < \infty$, the left side breaks apart, and we get

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} A_i\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) \\ &= \sum_{i=1}^{\infty} \mu(A_i). \end{aligned}$$

Bass 3.3 First observe that $\hat{A}$ is a σ-algebra. Certainly $\mu(\phi) = 0$, since ϕ is countable. Now let $(A_i)_{i=1}^{\infty}$ be a sequence of pairwise-disjoint elements of $\hat{A}$. If all of the A_i are countable, then $\bigcup_{i=1}^{\infty} A_i$ is countable, so $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty}$ [redacted] $\mu(A_i)$. Now suppose A_j is uncountable for some $j \in \mathbb{N}$. Then A_j^c is countable, so $1 = \sum_{i=1}^{j-1} \mu(A_i) + \mu(A_j) + \sum_{i=j+1}^{\infty} \mu(A_i)$, since A_j is uncountable and $A_i \subset A_j^c \ \forall\, i \neq j$. Also $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1$, since the set $\bigcup_{i=1}^{\infty} A_i$ is uncountable.

Bass 3.4 By countable additivity, we have that

$$\mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A) + \mu(B \cap A)$$

is equal to $\mu(A) + \mu(B)$, and also to

$$\mu(A \cup B) + \mu(A \cap B).$$

Bass 3.6 The point of this exercise is that we may define new measures by restricting μ. Fix $B \in \hat{A}$. We have $\nu(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$. Now let (A_i) be a sequence of pairwise disjoint elements of $\hat{A}$. Then

$$\begin{aligned} \nu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(\left[\bigcup_{i=1}^{\infty} A_i\right] \cap B\right) \\ &= \mu\left(\bigcup_{i=1}^{\infty} B \cap A_i\right) \\ &= \sum_{i=1}^{\infty} \mu(A_i \cap B) \\ &= \sum_{i=1}^{\infty} \nu(A_i). \end{aligned}$$

Bass 3.9 This problem is tricky, since we do not have a convenient way to represent an arbitrary Borel set. Instead, let's argue indirectly. First assume $m(\mathbb{R}) < \infty$ and $n(\mathbb{R}) < \infty$. Notice that $m(\mathbb{R}) = \lim_{k \to \infty} m((-k,k)) = \lim_{k\to\infty} n((-k,k)) = n(\mathbb{R})$.

Define $\hat{H} = \{ (a,b] : -\infty \leq a < b \leq \infty \}$. Since m and n agree on intervals (open, finite length), it follows that m and n agree on every set in $\hat{H}$. Next, m and n agree on the algebra of sets $\mathrm{Alg}(\hat{H})$ generated by $\hat{H}$. [Why?]

Now define $\hat{D} = \{ A \in \hat{B} : m(A) = n(A) \}$. By the above, $\hat{D} \supset \mathrm{Alg}(\hat{H})$. But $\hat{D}$ is a monotone class! (Why?) By the monotone class theorem (Bass Thm. 2.10), $\hat{D} = \hat{B}$.

Now we treat the general case. Let $B \in \hat{B}$. Then

$$\begin{aligned} m(B) &= \lim_{k\to\infty} m(B \cap (-k,k)) \\ &= \lim_{k \to \infty} n(B \cap (-k,k)) = n(B). \end{aligned}$$

Here we use that the measures m_k and n_k defined by $m_k(B) = m(B \cap (-k,k))$ and $n_k(B) = n(B \cap (-k,k))$ are equal $\forall k \in \mathbb{N}$ by the above. $\square$