Math 6320 Fall 2019

Assignment 2 Solutions

4.1 The idea here is that if we start with a (Borel) measure μ, and perform the Lebesgue-Stieltjes procedure, we recover μ.

Solution. We first check that $\ell((a,b]) = \mu((a,b])$.

- $0 \leq a < b$: $\alpha(b) - \alpha(a)$
$$= \mu((0,b]) - \mu((0,a])$$
$$= \mu((a,b]).$$
- $a < 0 \leq b$: $\alpha(b) - \alpha(a)$
$$= \mu((0,b]) + \mu((a,0])$$
$$= \mu((a,b]).$$
- $a < b < 0$: $\alpha(b) - \alpha(a)$
$$= \mu((a,0]) - \mu((b,0])$$
$$= \mu((a,b]).$$

Now we show that the Lebesgue-Stieltjes measure corresponding to α, m, agrees with μ on Borel sets. Suppose A is Borel. Let $\varepsilon > 0$. $\exists$ half-open intervals $(c_i, d_i]$ such that
$$A \subset \bigcup_{i=1}^{\infty} (c_i, d_i] \text{ and } m\left(\bigcup_{i=1}^{\infty} (c_i, d_i]\right) = \sum_{i=1}^{\infty} m((c_i, d_i])$$
$\leq m(A) + \varepsilon$. (We may assume the intervals are pairwise-disjoint. But
$$\sum_{i=1}^{\infty} m((c_i,d_i]) = \sum_{i=1}^{\infty} \ell((c_i,d_i])$$
$$= \sum_{i=1}^{\infty} \mu((c_i,d_i])$$
$$= \mu\left(\bigcup_{i=1}^{\infty} (c_i,d_i]\right).$$

We conclude that $\mu(A) \leq m(A)$. Similarly, $\mu(A^c) \leq m(A^c)$.

If $\mu(\mathbb{R}) < \infty$: $\mu(A) + m(A) + \mu(A^c)$
$$\leq \mu(A) + m(A) + m(A^c)$$
$$\Rightarrow \mu(\mathbb{R}) + m(A) \leq m(\mathbb{R}) + \mu(A)$$
$$\Rightarrow m(A) \leq \mu(A).$$

So when $\mu(\mathbb{R}) < \infty$, we have shown that $\mu(A) = m(A)$ for all Borel sets A.

If $\mu(\mathbb{R}) = \infty$, then for $n \in \mathbb{N}$ define μ_n and m_n by $\mu_n(A) = \mu(A \cap (-n, n])$ and $m_n(A) = m(A \cap (-n, n])$. By previous, we have $\mu_n = m_n$ as Borel measures, so

$$\mu(A) = \lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} m_n(A) = m(A). \quad \square$$

$\boxed{4.3}$ The point of this problem is that if we start with a measure μ and perform the Caratheodory construction, we recover μ.

First we show that μ^* is an outer measure.

(1) $\mu^*(\emptyset) = 0$, since $\emptyset \in \hat{A}$ and $\mu(\emptyset) = 0$.

(2) For any $E, F \subset X$, $\mu^*(E) \leq \mu^*(F)$ by definition of μ^*.

(3) Let (E_i) be a sequence of subsets of X. If $\mu^*(E_i) = \infty$ for some $i \in \mathbb{N}$, countable subadditivity follows immediately. So assume $\mu^*(E_i) < \infty$ $\forall i$. Let $\varepsilon > 0$. For each i, $\exists$ $B_i \in \hat{A}$ such that $B_i \supset E_i$ and $\mu(B_i) < \mu^*(E_i) + \frac{\varepsilon}{2^i}$. Then $B = \bigcup_{i=1}^\infty B_i \in \hat{A}$

Then $B \supset \bigcup_{i=1}^\infty E_i$ and $\mu(B) \leq \sum_{i=1}^\infty \mu(B_i) < \left(\sum_{i=1}^\infty \mu^*(E_i)\right) + \varepsilon$.

We conclude that $\mu^*\left(\bigcup_{i=1}^\infty E_i\right) \leq \sum_{i=1}^\infty \mu^*(E_i)$.

Now let $A \in \hat{A}$. We will show that A is μ^*-measurable. Let $E \subset X$. We must show that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. This is trivial if $\mu^*(E) = \infty$, so assume $\mu^*(E) < \infty$. For $\varepsilon > 0$, $\exists$ $B \in \hat{A}$ such that $B \supset E$ and $\mu(B) < \mu^*(E) + \varepsilon$.

$$\text{Then } \mu(B) = \mu(B \cap A) + \mu(B \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since ε was arbitrary, $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. Finally, for $A \in \hat{A}$, $\mu^*(A) \leq \mu(A)$, and $\mu(A) \leq \mu^*(A)$, since $\mu(A) \leq \mu(B)$ for every $B \in \hat{A}$ such that $B \supset A$. $\square$

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$\boxed{4.5}$ (comments) The equality $m(x+A) = m(A)$ refers to the fact that Lebesgue measure is translation-invariant. It follows directly from the fact the function ℓ (see Bass pg. 28), in the specific case of Lebesgue measure, satisfies $\ell((a+z, b+z]) = \ell((a,b])$. Similarly, for $c \geq 0$, $m(cA) = c\,m(A)$ follows directly from $\ell((ca, cb]) = c\,\ell((a,b])$.

$\boxed{4.6}$ (1) Measurability of B follows from the representation

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k .$$

(2) Assume $m(A_n) > \delta > 0$ for every $n \in \mathbb{N}$. Define $T_n = \bigcup_{k=n}^{\infty} A_k$. Then $T_1 \supset T_2 \supset \cdots$ and $T_1 \subset [0,1]$, so $m(B) = \lim_{n\to\infty} m(T_n) \geq \delta$.

(3) This is part of the Borel–Cantelli lemma!
Assume $\sum_{k=1}^{\infty} m(A_k) < \infty$. For every $n \in \mathbb{N}$, $m(T_n) \leq \sum_{k=n}^{\infty} m(A_k)$, and $\sum_{k=n}^{\infty} m(A_k) \to 0$ as $n \to \infty$. We conclude that $m(B) = \lim_{n\to\infty} m(T_n) = 0$.

(4) A simple example: $A_n = [0, \frac{1}{n}]$.

$\boxed{4.7}$ Let $\varepsilon \in (0,1)$. Define $E \subset [0,1]$ by

$$E = \{x \in \mathbb{Q} : 0 \leq x \leq 1\} \cup (0, \varepsilon).$$

By the density of rationals in the reals, $\overline{E} = [0,1]$.
Next,

$$\begin{aligned} m(E) &\leq m(\mathbb{Q} \cap [0,1]) + m((0,\varepsilon)) \\ &= 0 + m((0,\varepsilon)) \\ &= \varepsilon. \end{aligned}$$

On the other hand, $m(E) \geq m((0,\varepsilon)) = \varepsilon$. We conclude that $m(E) = \varepsilon$. $\square$

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$\boxed{4.10}$ Fix $\varepsilon \in (0,1)$. Let $A \subset \mathbb{R}$ be Borel measurable. Assume $m(A \cap I) \leq (1-\varepsilon)\, m(I)$ for every interval I. We claim that $m(A) = 0$. First assume $m(A) < \infty$. By outer regularity, $\exists$ a collection (c_i, d_i) of pairwise-disjoint open intervals such that $\bigcup_{i=1}^{\infty} (c_i, d_i) \supset A$ and $\sum_{i=1}^{\infty} (d_i - c_i) < m(A) + \delta$. (Here $\delta > 0$.)

But then
$$
\begin{aligned}
m(A) &= \sum_{i=1}^{\infty} m(A \cap (c_i, d_i)) \\
&\leq \sum_{i=1}^{\infty} (1-\varepsilon)(d_i - c_i) \\
&< (1-\varepsilon)[m(A) + \delta].
\end{aligned}
$$
Since $\delta > 0$ was arbitrary, we have $m(A) \leq (1-\varepsilon) m(A)$. This forces $m(A) = 0$.

Now, for the general case, let $A_n = A \cap [-n, n]$. Since $m(A_n) < \infty$, the above argument yields $m(A_n) = 0$. But then $m(A) = \lim_{n\to\infty} m(A_n) = 0$. $\square$

$\boxed{4.11}$ (Steinhaus theorem)

The idea here is to think in terms of translating sets. Suppose $A \subset \mathbb{R}$ is a Borel set and let $z \in \mathbb{R}$. Consider the intersection $A \cap (A + z)$, where $A + z = \{a + z : a \in A\}$. If this intersection is nonempty, say $w \in A \cap (A + z)$, then $w = a_1$ and $w = a_2 + z$ for some $a_1, a_2 \in A$. This implies that $z = a_1 - a_2$, so z is in the difference set B!

So now suppose $m(A) > 0$. By Exercise 4.10, $\exists$ an interval $I = (c,d)$ of finite length such that
$$m(A \cap I) > \tfrac{3}{4} m(I).$$
Set $\delta = \frac{1}{2}(d - c)$. For any $z \in (-\delta, \delta)$,
$$(A \cap I) \cap ((A \cap I) + z) \neq \emptyset. \quad [\text{Why?}]$$
By the reasoning above, the difference set B contains $(-\delta, \delta)$. $\square$

$\boxed{4.14}$ Here is a fun argument using Steinhaus. Define $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x - y \in \mathbb{Q}$. This equivalence relation induces a partition of $\mathbb{R}$ into equivalence classes. Let Γ be a set that contains exactly one element from each equivalence class (we invoke the axiom of choice here). Observe that

$$\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (\Gamma + q),$$

and that the sets in this union are pairwise disjoint.

Now let A be a Lebesgue measurable set and assume $m(A) > 0$. If $A \cap (\Gamma + q)$ were not Lebesgue measurable for some $q \in \mathbb{Q}$, we would be done.

So assume $A \cap (\Gamma + q)$ is Lebesgue measurable for all $q \in \mathbb{Q}$. Let $q \in \mathbb{Q}$. We have

$$\begin{aligned} \big(A \cap (\Gamma + q)\big) - \big(A \cap (\Gamma + q)\big) &\subseteq \big((\Gamma + q) - (\Gamma + q)\big) \\ &= \Gamma - \Gamma. \end{aligned}$$

But $\Gamma - \Gamma$ contains no open interval around zero [WHY], so neither does

$$\big(A \cap (\Gamma + q)\big) - \big(A \cap (\Gamma + q)\big).$$

By Steinhaus,

$$m\big(A \cap (\Gamma + q)\big) = 0.$$

But then

$$A = \bigcup_{q \in \mathbb{Q}} A \cap (\Gamma + q), \quad \text{and the union}$$

is countable, so $m(A) = 0$. Contradiction.