Math 6320 Fall 2019
Assignment 3 Solutions

$\boxed{\text{Bass 5.1}}$ Let $z \in \mathbb{R}$. We have

$$(*) \qquad \{ x : f(x) > z \} = \bigcup_{\substack{q \in \mathbb{Q} \\ q > z}} \{ x : f(x) > q \}.$$

By assumption, for each rational q, $\{x : f(x) > q\} \in \hat{A}$. Since the union on the right side of $(*)$ is a countable union, we conclude that $\{ x : f(x) > z \} \in \hat{A}$. By Proposition 5.5 of Bass, f is measurable. □

$\boxed{\text{Bass 5.4}}$ The idea here is to express A in a nice way. We will use the fact that a sequence of real numbers converges iff it is Cauchy.

Step 1: For $n, m, k \in \mathbb{N}$, define

$$G_{n,m,k} = \{ x : | f_n(x) - f_m(x) | \leq \tfrac{1}{k} \}.$$

Since $f_n - f_m$ is measurable and $|\cdot|$ is continuous, $|f_n(\cdot) - f_m(\cdot)|$ is a measurable function, so $G_{n,m,k}$ is a measurable set.

Step 2: For $N, k \in \mathbb{N}$, define

$$T_{N,k} = \{ x : |f_n(x) - f_m(x)| \leq \tfrac{1}{k} \text{ for all } n,m \geq N \}.$$

Since $T_{N,k} = \bigcap_{\substack{n,m \in \mathbb{N} \\ n,m \geq N}} G_{n,m,k}$, a countable intersection, it follows that $T_{N,k}$ is measurable.

Step 3: Define $S_k = \{ x : \exists\, N \in \mathbb{N} \text{ for which } |f_n(x) - f_m(x)| \leq \tfrac{1}{k}\ \forall n,m \geq N \}$.

Then $S_k = \bigcup_{N \in \mathbb{N}} T_{N,k}$, a measurable set.

Finally, $A = \bigcap_{k=1}^{\infty} S_k$, a measurable set. □

$\boxed{\text{Bass 5.9}}$ First assume g is Borel measurable (this includes continuous g). Let $a \in \mathbb{R}$. Then $(g \circ f)^{-1}((a, \infty))$
$$= f^{-1}(g^{-1}((a, \infty))).$$
Now $g^{-1}((a,\infty))$ is a Borel set, so the preimage of this set under f is measurable (Lebesgue meas.) by Proposition 5.11 of Bass.

Interestingly, the composition of Lebesgue meas. functions need not be Lebesgue measurable. For example, let $\varphi : [0,1] \to \mathbb{R}$ be the Cantor function. We argued in class that $h : [0,1] \to \mathbb{R}$ defined by $h(x) = x + \varphi(x)$ is continuous, maps $[0,1]$ invertibly onto $[0,2]$, and when viewed as $h : [0,1] \to [0,2]$, h^{-1} is continuous. Let $C \subset [0,1]$ denote the Cantor set. Since $m(h(C)) > 0$ $(m(h(C)) = 1)$, $h(C)$ has non-measurable subsets; let N be such a set. Let $E = h^{-1}(N)$.

Since $E \subset C$ and $m(C) = 0$, E is Leb. measurable since Lebesgue measure is complete. Consequently, χ_E, the characteristic function of E, is Lebesgue measurable. h^{-1} is continuous, so h^{-1} is Lebesgue measurable. But
$$\chi_E \circ h^{-1} = \chi_N$$
and χ_N is not Lebesgue measurable!

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$\boxed{\text{Bass 6.2}}$ First, f is automatically $\hat{\mathcal{A}}$-meas., since $\hat{\mathcal{A}}$ is the power set. Assume WLOG that $f(y) \geq 0$. Let $s = \sum_{i=1}^{n} a_i \chi_{E_i}$ be a simple function such that $0 \leq s \leq f^+$. Assume s is in canonical form, so in particular the E_i are pair-wise disjoint. Now

$$s(y) = \begin{cases} 0, & \text{if } y \notin E_i \ \forall\, 1 \leq i \leq n; \\ a_j, & \text{if } y \in E_j. \end{cases}$$

Either way, $\int_X s \, d\delta_y = s(y) \leq f^+(y) = f(y)$.

Since s was arbitrary, $\int_X f^+ d\delta_y \leq f(y)$. On the other hand, $\hat{s} = f(y) \chi_{\{y\}}$ is a simple function satisfying $0 \leq \hat{s} \leq f^+$, so

$$\int_X f^+ d\delta_y \geq \int_X \hat{s} \, d\delta_y = f(y).$$

We conclude that $\int_X f^+ d\delta_y = f(y)$. A similar argument gives $\int_X f^- d\delta_y = 0$. We conclude that

$$\int_X f \, d\delta_y = \int_X f^+ d\delta_y - \int_X f^- d\delta_y = f(y). \ \square$$

$\boxed{\text{Bass 6.4}}$ Since μ is σ-finite, there exists a sequence of measurable sets $(F_i)_{i=1}^{\infty}$ such that $F_1 \subset F_2 \subset \cdots$, $\mu(F_i) < \infty \ \forall i$, and $X = \bigcup_{i=1}^{\infty} F_i$. To find the desired sequence of simple functions, just start with the sequence (s_n) from Prop. 5.14 and multiply by χ_{F_n}: $s_n \chi_{F_n}$. $\square$

Bass 6.6 This is known as absolute continuity of the integral. The σ-finite hypothesis is not needed! The result is automatic if f is bounded, so let us work with "bounded approximations". Let A_ℓ, for $\ell \in \mathbb{N}$, be the set $A_\ell = \{x \in X : |f(x)| > \ell\}$. We want to control $\int_{A_\ell} |f| \, d\mu$. In class we used the Lebesgue dominated convergence theorem to do so. Here is a more elementary approach:

Define ν on $\hat{A}$ by $\nu(E) = \int_E |f| \, d\mu$. Then ν is a measure. Moreover, ν is a finite measure since f is integrable. The sets A_ℓ satisfy $A_1 \supset A_2 \supset \cdots$, and $\mu\left(\bigcap_{\ell=1}^\infty A_\ell\right) = 0$, since f integrable implies that f is finite almost everywhere with respect to μ. So therefore $\nu\left(\bigcap_{\ell=1}^\infty A_\ell\right) = 0$ as well, and since ν is a finite measure,

$$\lim_{\ell \to \infty} \nu(A_\ell) = 0.$$

Now let $\varepsilon > 0$. Find L such that $\nu(A_L) < \varepsilon/2$. Set $\delta = \varepsilon/2L$. Let $A \in \hat{A}$ satisfy $\mu(A) < \delta$. Then

$$\begin{aligned}
\int_A |f(x)| \, d\mu(x) &= \int_{A \cap A_L} |f(x)| \, d\mu(x) \\
&\quad + \int_{A \cap A_L^c} |f(x)| \, d\mu(x) \\
&\leq \nu(A_L) + L\mu(A) \\
&< \varepsilon/2 + L\left(\varepsilon/2L\right) \\
&= \varepsilon. \quad \square
\end{aligned}$$

$\boxed{\text{Bass 7.3}}$ Apply the Lebesgue dominated convergence theorem to the sequence $(f\chi_{A_n})$, and $f\chi_A$.

$\boxed{\text{Bass 7.4}}$ Suppose $(X, \mathcal{A}, \mu)$ is a measure space and (f_n) is a sequence of measurable functions such that
$$\sum_{n=1}^{\infty} \int_X |f_n|\, d\mu < \infty.$$
Using Prop. 7.6 of Bass, $\int_X \sum_{n=1}^{\infty} |f_n(x)|\, d\mu(x) = \sum_{n=1}^{\infty} \int_X |f_n|\, d\mu < \infty$. The function in $\int_X \sum_{n=1}^{\infty} |f_n(x)|\, d\mu(x)$, namely, $\sum_{n=1}^{\infty} |f_n(x)|$, is therefore finite for μ-a.e. $x \in X$. Consequently, $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges absolutely, and therefore converges, for μ-a.e. $x \in X$.

Finally, for every $N \in \mathbb{N}$ we have
$$\left| \sum_{n=1}^{N} f_n(x) \right| \leq \sum_{n=1}^{N} |f_n(x)| \leq \sum_{n=1}^{\infty} |f_n(x)|,$$
and the latter is integrable, so by LDCT we have
$$\int_X f(x)\, d\mu(x) = \int_X \lim_{N \to \infty} \sum_{n=1}^{N} f_n(x)\, d\mu(x)$$
$$= \lim_{N \to \infty} \int_X \sum_{n=1}^{N} f_n(x)\, d\mu(x)$$
$$= \sum_{n=1}^{\infty} \int_X f_n(x)\, d\mu(x). \quad \square$$

$\boxed{\text{Bass 7.5}}$ We have $0 \leq g_n + f_n$ and $0 \leq g_n - f_n$, so Fatou gives
$$\int g + f\, d\mu \leq \liminf_n \int g_n + f_n\, d\mu$$
$$= \int g\, d\mu + \liminf_n \int f_n\, d\mu.$$
Also, $\int g - f\, d\mu \leq \liminf_n \int g_n - f_n\, d\mu = \int g\, d\mu - \limsup_n \int f_n\, d\mu.$

Consequently, $\int f \, d\mu \leq \liminf_n \int f_n \, d\mu$

$$\leq \limsup_n \int f_n \, d\mu$$

$$\leq \int f \, d\mu,$$

so $\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu$. $\square$

$\boxed{\text{Bass 7.6}}$ We work on $\mathbb{R}$ with Lebesgue measure m.

Example: $f_n = \frac{1}{n} \chi_{[n, n+1]}$.