Math 6320 Fall 2019
Assignment 4 Solutions

$\boxed{7.7}$ Suppose $(X, \hat{A}, \mu)$ is a measure space, $f_n : X \to \mathbb{R}$ is a sequence of non-negative integrable functions, $f : X \to \mathbb{R}$ is non-negative integrable, $f_n \to f$ a.e., and $\int_X f_n \, d\mu \to \int_X f \, d\mu$.

Let $A \in \hat{A}$. We will prove that $\int_A f_n \, d\mu \to \int_A f \, d\mu$. The idea is to apply Fatou to the sequences

$$f_n - f_n \chi_A$$
$$f_n + f_n \chi_A,$$

both of which consist of non-negative functions. For the first sequence, Fatou yields

$$\int_X (f - f\chi_A) \, d\mu \le \underline{\lim} \int_X (f_n - f_n \chi_A) \, d\mu$$
$$= \int_X f \, d\mu - \overline{\lim} \int_X f_n \chi_A \, d\mu .$$

Since f is integrable, we obtain

$$(*) \qquad \overline{\lim} \int_X f_n \chi_A \, d\mu \le \int_X f \chi_A \, d\mu .$$

For the second sequence, Fatou yields

$$(**) \qquad \int_X f \chi_A \, d\mu \le \underline{\lim} \int_X f_n \chi_A \, d\mu .$$

$(*)$ and $(**)$ imply that $\int_X f_n \chi_A \, d\mu$ converges, and converges to $\int_X f \chi_A \, d\mu$. □

$\boxed{7.8}$ This is a direct application of the generalized Lebesgue DCT. Consider the sequence $|f_n - f|$. We have $|f_n - f| \to 0$ a.e., $|f_n - f| \le g_n = |f_n| + |f|$, $g_n \to 2|f|$ a.e., $\int_X g_n \, d\mu \to 2 \int_X |f| \, d\mu$. By the generalized LDCT, $\int_X |f_n - f| \to 0$. □

$\boxed{7.10}$ This is not true in general. For instance, let $f_n = \chi_{[n,n+1]}$. Then $\int_{\mathbb{R}} f_n \, dm = 1$ for every $n \in \mathbb{N}$, so $\overline{\lim} \int_{\mathbb{R}} f_n \, dm = 1$. On the other hand, $f_n(x) \to 0$ as $n \to \infty$ for every $x \in \mathbb{R}$, so

$$\int_{\mathbb{R}} (\overline{\lim}\, f_n)(x) \, dm(x) = \int_{\mathbb{R}} 0 \, dm = 0.$$

$\boxed{7.12}$ The limit is equivalent to

$$\lim_{n\to\infty} \int_{[0,\infty)} (1 - x/n)^n \log(2 + \cos(x/n)) \overset{\chi_{[0,n]}(x)}{\downarrow} dm(x).$$

Convergence follows from the monotone convergence theorem. First, $(1 - x/n)^n \uparrow e^{-x}$ for every $x \geq 0$. Second, $\cos(x/n) \uparrow 1$ for every $x \in [0,n]$. The integrand is a non-negative function for every n, so MCT applies. The limit is

$$\log(3) \int_{[0,\infty)} e^{-x} \, dm(x) = \log(3). \quad \square$$

$\boxed{7.14}$ Consider

$$(*) \quad \lim_{n\to\infty} \int_0^\infty n e^{-nx} \left(\frac{x^2+1}{x^2+x+1} \right) dx.$$

Let $r(x) = (x^2+1)/(x^2+x+1)$. Observe that $r(x) \in (0,1]$ for all $x \geq 0$, and $r(0) = 1$. The pointwise limit is interesting:

$$\lim_{n\to\infty} n e^{-nx} r(x) = \begin{cases} \infty, & \text{if } x = 0; \\ 0, & \text{if } x > 0. \end{cases}$$

This is a "concentration of mass" problem: As n grows, the mass under $\underset{= 1 \ \forall n \ !!}{\underset{\downarrow}{n e^{-nx}}}$ concentrates near $x=0$.

Since r is continuous at zero, we therefore expect the limit in $(*)$ to equal $r(0) = 1$. This follows rigorously from the following two observations.

① For each fixed $\hat{x} > 0$,

$$\lim_{n\to\infty} \int_{\hat{x}}^{\infty} n e^{-nx} r(x)\, dx = 0.$$

To see this, note that $\partial/\partial t\,(te^{-tx}) = e^{-tx}(1-tx)$, so $n \geq N \doteq \lceil 1/\hat{x} \rceil$, we can bound the integrand by the integrable function Ne^{-Nx} and therefore apply the Lebesgue DCT.

② Since r is continuous at 0, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $1-\varepsilon \leq r(x) \leq 1$ for all $x \in [0,\delta]$. This implies

$$\int_0^\delta n e^{-nx} r(x)\, dx \quad \text{satisfies}$$

$$\underbrace{(1-\varepsilon)\int_0^\delta n e^{-nx}\, dx}_{=\,(1-\varepsilon)(1-e^{-n\delta})} \leq \int_0^\delta n e^{-nx} r(x)\, dx \leq \underbrace{\int_0^\delta n e^{-nx}\, dx}_{=\,1-e^{-n\delta}}.$$

7.15 The limit is equivalent to

$$(*) \qquad \lim_{n\to\infty} \int_{\mathbb{R}} f(1 + x/n^2)\, g(x)\, \chi_{[-n,n]}(x)\, dm(x).$$

The pointwise limit of the integrand is $f(1)g(x)$. Since f is bounded, say by M, the absolute value of the integrand is bounded by $M|g(x)|$, and this function is integrable. The Lebesgue DCT therefore applies, and

$$(*) = \int_{\mathbb{R}} f(1)\, g(x)\, dm(x)$$

$$= f(1) \int_{\mathbb{R}} g(x)\, dm(x).$$

$\boxed{17.19}$ Note: It is more common to say that (f_n) is uniformly absolutely continuous if for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that if $A \in \hat{A}$ satisfies $\mu(A) < \delta$, then $\int_A |f_n|\,d\mu < \varepsilon$ for every $n \in \mathbb{N}$. I will use this definition here.

Suppose $\mu(X) < \infty$. We show that (f_n) is UI iff $\sup_n \int_X |f_n|\,d\mu < \infty$ and (f_n) is UAC.

Proof: ($\Rightarrow$) Suppose (f_n) is UI. Using the definition of UI with $\varepsilon = 1$, we have

$$\int_X |f_n|\,d\mu = \int_{\{x : |f_n(x)| > M\}} |f_n(x)|\,d\mu(x) + \int_{\{x : |f_n(x)| \le M\}} |f_n(x)|\,d\mu(x)$$

$$< \underbrace{1 + M\mu(X)}_{\text{independent of } n}.$$

UI $\Rightarrow$ UAC: Let $\varepsilon > 0$. $\exists\ L$ such that for every n,

$$\int_{\{x : |f_n(x)| > L\}} |f_n(x)|\,d\mu(x) < \varepsilon/2.$$

Now set $\delta = \varepsilon/2L$. If $A \in \hat{A}$ satisfies $\mu(A) < \delta$, then

$$\int_A |f_n(x)|\,d\mu(x) = \int_{A \cap \{x : |f_n(x)| > L\}} |f_n(x)|\,d\mu(x) + \int_{A \cap \{x : |f_n(x)| \le L\}} |f_n(x)|\,d\mu(x)$$

$$< \varepsilon/2 + L(\varepsilon/2L)$$

$$= \varepsilon,$$

for every $n \in \mathbb{N}$.

($\Leftarrow$) Suppose $S = \sup_n \int_X |f_n|\,d\mu < \infty$ and suppose that (f_n) is UAC. By Chebyshev inequality (Lemma 10.4), it follows that for every $a > 0$ and $n \in \mathbb{N}$,

$$\mu(\{x : |f_n(x)| > a\}) \le S/a.$$

In particular,

$$\mu(\{x : |f_n(x)| > M\}) \to 0 \text{ as } M \to \infty,$$

uniformly in n. This fact, together with the assumption that (f_n) is UAC, implies that (f_n) is UI. $\square$

|Bass 8.1| Suppose $f : X \to [0, \infty]$ is measurable. We show that if $\int_X f \, d\mu < \infty$, then $f < \infty$ a.e.

Proof: Let $S_n = \{x : f(x) \geq n\}$. Observe that $S_1 \supset S_2 \supset S_3 \supset \cdots$, and

$$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} S_n.$$

Since $\mu(S_1) < \infty$ and the S_n are nested downward, $\mu(\bigcap_{n=1}^{\infty} S_n) = \lim_{n\to\infty} \mu(S_n)$. By Chebyshev, $\mu(S_n) \leq \int_X f \, d\mu / n$, so $\lim_{n\to\infty} \mu(S_n) = 0$.

Conclusion:

$$\mu(\cap_{n=1}^{\infty} S_n) = 0.$$

|8.5| Define the measure ν on $\hat{A}$ by $\nu(A) = \int_A f \, d\mu$. Since f is integrable, ν is a finite measure. Let (t_n) be any increasing sequence of positive reals such that $t_n \to \infty$ as $n \to \infty$. For each n, we have

$$t_n \mu \underbrace{\{x : f(x) \geq t_n\}}_{E_{t_n}} \leq \nu(E_{t_n}).$$

Next, $E_{t_1} \supset E_{t_2} \supset \cdots$, so $\nu(E_{t_n}) \xrightarrow{n\to\infty} \nu\left(\bigcap_{n=1}^{\infty} E_{t_n}\right)$.

Finally, $\mu(\cap_{n=1}^{\infty} E_{t_n}) = 0$, so $\nu(\cap_{n=1}^{\infty} E_{t_n}) = 0$.

Since (t_n) was arbitrary, the proof is complete.

|8.13| First prove this for a continuous function with compact support, and then use Theorem 8.4.