Math 6320 Fall 2019

Assignment 5 Solutions

$\boxed{\text{Bass 10.1}}$ This result appears in Folland as Theorem 2.30. The proof uses the "fast subsequence" idea. Since (f_n) is Cauchy in measure, $\exists$ a subsequence $(g_j) = (f_{n_j})$ such that if $E_j = \{x : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}$, then $\mu(E_j) \leq 2^{-j}$. Set $F_k = \bigcup_{j=k}^{\infty} E_j$.

Then $\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k}$.

If $x \notin F_k$, then for $i \geq j \geq k$, we have

$$|g_j(x) - g_i(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \qquad (*)$$
$$\leq \sum_{l=j}^{i-1} 2^{-l} \leq 2^{1-j}.$$

Set $F = \bigcap_{k=1}^{\infty} F_k = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$. Observe that $\mu(F)=0$, and (g_j) is pointwise Cauchy on $X \setminus F$. Define

$$f(x) = \begin{cases} \lim g_j(x), & x \notin F; \\ 0, & x \in F \end{cases}.$$

Then f is measurable and $g_j(x) \to f(x)$ for μ-a.e. $x \in X$. Next, $(*)$ implies

$$|g_j(x) - f(x)| \leq 2^{1-j}$$

for $x \notin F_k$, $j \geq k$. Since $\mu(F_k) \to 0$ as $k \to \infty$, it follows that $g_j \to f$ in measure. Finally, we return to the original sequence: $f_n \to f$ in measure, since

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$
$$\subset \{x : |f_n(x) - g_j(x)| \geq \varepsilon/2\}$$
$$\cup \{x : |g_j(x) - f(x)| \geq \varepsilon/2\},$$

and these two sets have small measure for large n, j.

$\boxed{\text{Bass } 10.2}$ First we discuss the properties of d.

(1) $d(f,g) \geq 0$, and $d(f,g) = 0$ iff $f = g$ a.e.

(2) d is symmetric by definition

(3) We obtain the triangle inequality by arguing pointwise. For $x \in X$, we have

$$\frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \leq \frac{[\,|f(x)-k(x)| + |k(x)-g(x)|\,]}{1 + [\,|f(x)-k(x)| + |k(x)-g(x)|\,]}$$

$$\leq \frac{|f(x)-k(x)|}{1+|f(x)-k(x)|} + \frac{|k(x)-g(x)|}{1+|k(x)-g(x)|}.$$

The first inequality holds because $t \to \frac{t}{1+t}$ is strictly increasing on $[0,\infty)$.

We prove that $f_n \to f$ in measure iff $d(f_n,f) \to 0$.

$(\Rightarrow)$
$$d(f_n,f) = \int_X \frac{|f_n - f|}{1+|f_n-f|}\,d\mu$$
$$\leq \int_{\{x:|f_n(x)-f(x)|>\varepsilon\}} \frac{|f_n-f|}{1+|f_n-f|}\,d\mu + \int_{\{x:|f_n(x)-f(x)|\leq\varepsilon\}} \frac{|f_n-f|}{1+|f_n-f|}\,d\mu$$
$$\leq \mu\{x : |f_n(x) - f(x)| > \varepsilon\} + \varepsilon\mu(X).$$

Here we use $|x|/(1+|x|) \leq \min(1,|x|)$.

$(\Leftarrow)$ For $\varepsilon > 0$, define $l_\varepsilon = \inf_{|x|\geq\varepsilon} |x|/(1+|x|) > 0$.

Then
$$\mu\{x\in X : |f_n(x) - f(x)| > \varepsilon\} = \frac{1}{l_\varepsilon}\int_{|f_n-f|>\varepsilon} l_\varepsilon\,d\mu$$
$$\leq \frac{1}{l_\varepsilon}\int_{\{x:|f_n(x)-f(x)|>\varepsilon\}} \frac{|f_n(x)-f(x)|}{1+|f_n(x)-f(x)|}\,d\mu(x)$$
$$\leq \frac{1}{l_\varepsilon}\, d(f_n,f).$$

| Bass 10.3 | This is a "convergence in measure" version of Fatou. The idea of the proof: Proceed in two steps.

Assume $f_n \geq 0$ and $f_n \to f$ in measure. Choose a subsequence (f_{n_i}) such that $\int_X f_{n_i}\,d\mu \xrightarrow{i} \liminf_{n\to\infty} \int f_n$.

Next, $f_{n_i} \to f$ in measure. Therefore, $\exists$ a further subsequence $f_{n_{i_j}}$ such that $f_{n_{i_j}} \to f$ a.e. Applying Fatou gives

$$\int_X f\,d\mu \leq \liminf_{j\to\infty} \int f_{n_{i_j}} = \liminf_{n\to\infty} \int f_n . \quad \square$$

| Bass 10.5 | This problem asserts that "almost uniform" convergence implies a.e. pointwise convergence.

Proof. For each $k \in \mathbb{N}$, $\exists$ F_k measurable such that $\mu(F_k^c) < \frac{1}{k}$ and $f_n \to f$ uniformly on F_k. Let $G = \bigcup_{k=1}^{\infty} F_k$. Then $X \setminus G = \bigcap_{k=1}^{\infty} X\setminus F_k$, so $\mu(X\setminus G) = 0$. Therefore $x \in G$ for μ-a.e. $x \in X$. If $x \in G$, then $x \in F_k$ for some k, and so $f_n(x) \to f(x)$, since $f_n \to f$ uniformly on F_k.

| Bass 10.6 |

Proof. Set $\varepsilon = 1/k$. Applying Borel–Cantelli, the set $B_k = \{ x : |f_n(x) - f(x)| > \frac{1}{k}$ for ∞ly many values of $n \}$ satisfies $\mu(B_k) = 0$. Set $B = \bigcup_{k=1}^{\infty} B_k$. Then

$\mu(B) = 0$, and $f_n(x) \to f(x)$ for every $x \in X \setminus B$.

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Bass 11.5

$$\int_{\mathbb{R}} |f(x)|\, dm(x)$$
$$= \int_{\mathbb{R}} \int_{\mathbb{R}_{\geq 0}} \mathbb{1}_{[0,|f(x)|]}(t)\, dm(t)\, dm(x)$$
$$\underset{\text{Tonelli}}{=} \int_{\mathbb{R}_{\geq 0}} \int_{\mathbb{R}} \mathbb{1}_{[0,|f(x)|]}(t)\, dm(x)\, dm(t)$$
$$= \int_{\mathbb{R}_{\geq 0}} m\{x \in \mathbb{R} : |f(x)| \geq t\}\, dm(t).$$

This is known as the layer cake representation of the integral.

Bass 11.6 By Tonelli,

$$1 = m_2(A) = \int_{[0,1]} \int_{[0,1]} \chi_A(x,y)\, dm(y)\, dm(x)$$
$$= \int_{[0,1]} m(s_x(A))\, dm(x).$$

Since $m(s_x(A)) \in [0,1]\ \forall x \in [0,1]$, and this integral gives one, we must have $m(s_x(A)) = 1$ for m-a.e. $x \in [0,1]$.

Bass 11.10

(1) It can be shown that $\mathcal{B} \otimes \mathcal{B} = \mathcal{B}_2$, where $\mathcal{B}_2$ is the Borel σ-algebra on $[0,1] \times [0,1]$ (see Folland). D, a closed set, is therefore a Borel set ($\mathcal{B}_2$ measurable), hence $\mathcal{B} \otimes \mathcal{B}$ measurable.

(2) $$\int_X \int_Y \chi_D(x,y)\, d\mu(y)\, dm(x) = 1$$

$$\int_Y \int_X \chi_D(x,y)\, dm(x)\, d\mu(y) = 0$$

There exists, because μ is not σ-finite, no contradiction.

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$\boxed{\text{Bass 11.14}}$

(A) $\displaystyle\int_0^b \int_0^\infty e^{-xy}\sin(x)\,dy\,dx$

$$= \int_0^b \frac{\sin(x)}{x}\,dx.$$

(B) $\displaystyle\int_0^\infty \int_0^b e^{-xy}\sin(x)\,dx\,dy$

$$= \int_0^\infty \left[\frac{e^{-xy}[-y\sin(x) - \cos(x)]}{1+y^2} \right]_0^b dy$$

$$= \int_0^\infty \left[\frac{e^{-by}[-y\sin(b) - \cos(b)]}{1+y^2} \right] + \left[\frac{1}{1+y^2} \right] dy$$

Applying dominated convergence, the integral in (B) converges to $\displaystyle\int_0^\infty \frac{1}{1+y^2}\,dy = \pi/2$ as $b \to \infty$.

By Fubini, which applies because $e^{-xy}\sin(x)$ is integrable wrt the product measure on $[0,b] \times [0,\infty)$, the integrals in (A) and (B) are equal, so we conclude

$$\lim_{b\to\infty} \int_0^b \frac{\sin(x)}{x}\,dx = \pi/2.$$

$\boxed{\text{Bass 11.16}}$ We have

$$\int_{\mathbb{R}} \sum_{n=1}^\infty \frac{|a_n|}{\sqrt{|x-r_n|}}\,dm(x) \qquad (*)$$

$$= \sum_{n=1}^\infty \left[\int_{\mathbb{R}} \frac{|a_n|}{\sqrt{|x-r_n|}}\,dm(x) \right] \qquad \text{(Tonelli)}$$

$$= \sum_{n=1}^\infty |a_n| \left[\int_{\mathbb{R}} \frac{1}{\sqrt{|x|}}\,dm(x) \right] \qquad \text{(Lebesgue measure is translation-invar.)}$$

$$= \sum_{n=1}^\infty K|a_n| < \infty.$$

Here $K = \int_{\mathbb{R}} 1/\sqrt{|x|}\,dm(x)$. Since $(*) < \infty$, the series in the integrand converges for m-a.e. $x \in \mathbb{R}$.