Math 6320 Fall 2019
Assignment 6 Solutions

Bass 15.1 We show L^∞ is complete. Write $(X, \hat{A}, \mu)$ for the measure space. Assume (f_k) is Cauchy in L^∞. For every $m \in \mathbb{N}$, $\exists\, n \in \mathbb{N}$ such that
$$|f_j(x) - f_k(x)| < 1/m \quad (**)$$
for all $j, k \geq n$ and $x \in N_{j,k,m}^c$, where $N_{j,k,m}$ is a set of measure zero. Define $N = \bigcup N_{j,k,m}$, where the union is taken over $j, k, m \in \mathbb{N}$.(*) Then $\mu(N) = 0$, and $(f_k(x))$ is Cauchy in $\mathbb{R}$ $\forall\, x \in N^c$. Define $f : X \to \mathbb{R}$ measurable by
$$f(x) = \begin{cases} \lim_{k\to\infty} f_k(x), & x \in N^c; \\ 0, & x \in N. \end{cases}$$
Letting $k \to \infty$ in $(**)$, we see that for every $m \in \mathbb{N}$ $\exists\, n \in \mathbb{N}$ st
$$|f_j(x) - f(x)| \leq 1/m$$
for all $j \geq n$ and $x \in N^c$. It follows that $f \in L^\infty$ and $f_j \to f$ in L^∞ as $j \to \infty$. $\square$

(*) The union $N = \bigcup N_{j,k,m}$ is taken over $m \in \mathbb{N}$ and $j, k \geq n(m)$. $\square$

Bass 15.3 Let $p \in [1, \infty)$. We have
$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^p\,dx &= \int_{\mathbb{R}} \int_0^\infty p\,t^{p-1} \chi_{\{0 \leq t \leq |f(x)|\}}\,dt\,dx \\ &\underset{\text{(Tonelli)}}{=} \int_0^\infty \int_{\mathbb{R}} p\,t^{p-1} \chi_{\{0 \leq t \leq |f(x)|\}}\,dx\,dt \\ &= \int_0^\infty p\,t^{p-1}\, m\{x : |f(x)| \geq t\}\,dt.\end{aligned}$$

Bass 15.4 I will assume $f \in L^{p=\infty}$, so $\|f\|_\infty < \infty$. The result is automatic if $\|f\|_\infty = 0$, so assume $\|f\|_\infty > 0$. Let $\delta > 0$ be such that $\|f\|_\infty - \delta > 0$ and define
$$S_\delta = \{x : |f(x)| \geq \|f\|_\infty - \delta\}.$$

We have $\|f\|_p \geq \left[\int_{S_\delta} (\|f\|_\infty - \delta)^p dm \right]^{1/p}$

$$= (\|f\|_\infty - \delta)\, m(S_\delta)^{1/p},$$

for all $p \in [1,\infty)$. Note that $m(S_\delta) > 0$. This inequality implies $\liminf_{p\to\infty} \|f\|_p \geq \|f\|_\infty$.

For the other inequality, note that $f \in L^\infty \Rightarrow f \in L^p$ for all $p \in [1,\infty)$ since the measure space is finite. Fix $q \in [1,\infty)$. For every $p > q$, we have

$$\|f\|_p \leq \left[\int_{[0,1]} \|f\|_\infty^{p-q} |f(x)|^q dm \right]^{1/p}$$

$$= \|f\|_\infty^{(p-q)/p} \|f\|_q^{q/p}.$$

This implies $\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty$.

We have $\|f\|_\infty \leq \underline{\lim} \|f\|_p \leq \overline{\lim} \|f\|_p \leq \|f\|_\infty$. This implies $\lim_{p\to\infty} \|f\|_p$ exists and equals $\|f\|_\infty$.

Bass 15.6

Consider Lebesgue measure m on $\mathbb{R}$. Let $a \in [1,\infty)$.

(A) $\dfrac{1}{x^{1/a}} \chi_{[a,\infty)}$

This function is in L^p for $p \in (a, \infty]$, and no other values of p.

(B) Let $b \in (1,\infty)$.

$\dfrac{1}{x^{1/b}} \chi_{[0,1/2]}$

This function is in L^p iff $p \in [1,b)$.

Bass 15.7 This is an interpolation result. We shall interpolate between r, s and then apply Hölder. Define $\lambda \in (0,1)$ by

$$p^{-1} = \lambda r^{-1} + (1-\lambda) s^{-1}.$$

This gives $1 = \dfrac{\lambda p}{r} + \dfrac{(1-\lambda) p}{s}$.

We have
$$\int_X |f|^p \, d\mu$$
$$= \int_X |f|^{\lambda p} |f|^{(1-\lambda)p} \, d\mu$$
(Hölder)
$$\le \| |f|^{\lambda p} \|_{r/\lambda p} \; \| |f|^{(1-\lambda)p} \|_{\frac{s}{(1-\lambda)p}}$$
$$= \left[\int_X |f|^r \right]^{\lambda p/r} \left[\int_X |f|^s \right]^{\frac{(1-\lambda)p}{s}}$$
$$= \|f\|_r^{\lambda p} \, \|f\|_s^{(1-\lambda)p} . \quad \square$$

$\boxed{\text{Bass 15.8}}$ Assume $f_n \to f$ in L^p and $g \in L^q$, where p and q are conjugate exponents. We have
$$\left| \int_X (f_n g - fg) \, d\mu \right|$$
$$\le \int_X |f_n g - fg| \, d\mu$$
(Hölder)
$$\le \|f_n - f\|_p \, \|g\|_q .$$
The right side converges to 0 as $n \to \infty$ since $f_n \to f$ in L^p.

$\boxed{\text{Bass 15.9}}$

(1) Let $f \in L^2([0,1])$. We have
$$\left| \int_{[0,1]} f(x) g_n(x) \, dm(x) \right|$$
$$\le \int_{[0,1]} |f(x) g_n(x)| \, dm(x)$$
(Hölder)
$$\le \left[\int_{[0,1]} |f|^2 dm \right]^{1/2} \left[\int_{[0,1]} |g_n|^2 dm \right]^{1/2}$$
$$= \|f\|_2 \cdot \left(\tfrac{1}{\sqrt{n}}\right) .$$
Thus
$$\left| \int_{[0,1]} f(x) g_n(x) \, dm(x) \right| \to 0 \text{ as } n \to \infty .$$

(2) Define f as follows. For each $k \in \mathbb{N}$, take $h_k > 0$ so that

$$\int_{[\frac{1}{k+1}, \frac{1}{k}]} h_k \, dm = \frac{1}{k[\ln(k)]^2},$$

assuming $k \geq 2$. Define f by

$$f(x) = \sum_{k=2}^{\infty} h_k \chi_{[\frac{1}{k+1}, \frac{1}{k}]}(x).$$

Observe that

$$\int_{[0,1]} f \, dm = \sum_{k=2}^{\infty} \frac{1}{k[(\ln(k))]^2} < \infty.$$

So $f \in L^1$, but one can argue that $\int_{[0,1]} f g_n \, dm$ does not converge to zero.

Bass 15.12 First observe that Fatou implies $f \in L^p$, since

$$\int_X |f|^p \, d\mu \leq \liminf_{n \to \infty} \int_X |f_n|^p \, d\mu \leq \sup_n \|f_n\|_p^p < \infty.$$

However, the hypotheses do not allow us to infer that $f_n \to f$ in the L^p sense. Instead, we use a small set – large set decomposition. First assume the measure space is finite. Set $M = \sup_n \|f_n\|_p$. Let $\varepsilon > 0$. Since $g \in L^q$, $\int_X |g|^q < \infty$, so there exists $\delta > 0$ such that if $\mu(A) < \delta$, then $\int_A |g|^q \, d\mu < \varepsilon$. By Egorov, $\exists$ a measurable set B such that $\mu(B) < \delta$ and $f_n \to f$ uniformly on $X \setminus B$.

We have the estimate

$$\int_X |f_n - f| \cdot |g| \, d\mu$$

$$= \int_B |f_n - f| \cdot |g| \, d\mu + \int_{X \setminus B} |f_n - f| \cdot |g| \, d\mu .$$

The first integral is bounded above by

$$\| f_n - f \|_p \, \| g \chi_B \|_q \le [\| f_n \|_p + \| f \|_p] \cdot \| g \chi_B \|_q \le 2M \, \varepsilon^{1/q} .$$

The second integral is bounded above by

$$\| (f_n - f) \chi_{X \setminus B} \|_\infty \, \| g \|_1 ,$$

noting that $g \in L^q \Rightarrow g \in L^1$ since the measure space is finite. These estimates imply the result.

Exercise: Explain why we can reduce to the finite measure space case.

L^1 counterexample:

space $= \mathbb{R}$
measure $=$ Lebesgue measure m

$f_n = \chi_{[n, n+1)}$
$f \equiv 0$
$g \equiv 1$