Exam 1: Math 6320 Fall 2019
Professor William Ott

Exercise 1. **(15)** Let X be a set and let $\mathcal{M}$ be a σ-algebra of subsets of X.

(a) **(5)** Define: The function $f : X \to \mathbb{R}$ is ***measurable***. You may either state the original definition, or provide any condition that is equivalent to the original definition.

(b) **(10)** Suppose $f_n : X \to \mathbb{R}$ is measurable for every $n \in \mathbb{N}$. Define the function g on X by $g(x) = \sup_{n\in\mathbb{N}} f_n(x)$. Assuming $g(x) < \infty$ for every $x \in X$, prove that g is measurable.

Exercise 2. **(15)** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) **(5)** Define: μ is a ***measure***.

(b) **(10)** Let $(E_n)_{n=1}^{\infty}$ be a sequence of sets in $\mathcal{M}$ satisfying $E_n \subset E_{n+1}$ for every $n \in \mathbb{N}$. Prove that

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \lim_{n\to\infty} \mu(E_n).$$

Exercise 3. **(15)** Describe the construction of Lebesgue measure m on $\mathbb{R}$. You do not need to prove anything here. Rather, it suffices the highlight the important components of the construction. Your description should include the definition of the Lebesgue outer measure m^*, the definition of an m^*-measurable set, and a careful statement of the Carathéodory theorem.

Exercise 4. **(10)** Let $f : [0, 1] \to \mathbb{R}$ be a nonnegative function and suppose that f is integrable with respect to Lebesgue measure m. Prove that

$$\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}\, \mathrm{d}m(x) = m(\{x \in [0, 1] : f(x) > 0\}).$$

Exercise 5. **(15)** Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) **(5)** State the ***monotone convergence theorem***.

(b) **(10)** Let $(f_n : X \to \mathbb{R})$ be a sequence of integrable functions and suppose $f : X \to \mathbb{R}$ is integrable. Prove that if

$$\sum_{n=1}^{\infty} \int_X |f_n(x) - f(x)|\, \mathrm{d}\mu(x) < \infty,$$

then $f_n \to f$ almost everywhere.

Exercise 6. **(30)** Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $f_n : X \to \mathbb{R}$ is measurable for every $n \in \mathbb{N}$ and suppose $f : X \to \mathbb{R}$ is measurable.

(a) **(5)** Define: $f_n \to f$ ***pointwise almost everywhere***.

(b) **(5)** Define: $f_n \to f$ ***in measure***.

(c) **(10)** Suppose $f_n \to f$ in measure. Does it follow that $f_n \to f$ almost everywhere? Either prove this, or provide a counterexample.

(d) **(10)** Prove that if μ is a finite measure and if $f_n \to f$ almost everywhere, then $f_n \to f$ in measure.