**Exam 2: Math 6320 Fall 2019**
**Professor William Ott**

**Exercise 1.** **(10, 5 each)** Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{A}, \nu)$ be measure spaces.

**(a)** Define the product $\sigma$-algebra $\mathcal{M} \otimes \mathcal{A}$.

**(b)** Carefully state the Fubini-Tonelli theorem for $f(x, y) : X \times Y \to \mathbb{R}$.

You do not need to prove anything in this problem.

**Exercise 2.** **(10)** Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Prove that

$$\int_{\mathbb{R}} |f(x)| \,\mathrm{d}m(x) = \int_0^\infty m\{x \in \mathbb{R} : |f(x)| \geqslant t\} \,\mathrm{d}m(t).$$

This is known as the layer cake representation of the integral.

**Exercise 3.** **(15)** (On relationships between the $L^p$ spaces)

**(a)** **(10)** Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Suppose $1 \leqslant p < q < \infty$. Prove that if $f \in L^q(X, \mu)$, then $f \in L^p(X, \mu)$.

**(b)** **(5)** Does this result remain true if $\mu(X) = \infty$? If yes, prove it. If no, provide a counterexample.

**Exercise 4.** **(10)** Recall that the convolution of two measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y) g(y) \,\mathrm{d}m(y).$$

Prove that if $f \in L^1(\mathbb{R}, m)$ and $g \in L^1(\mathbb{R}, m)$, then

$$\|f * g\|_1 \leqslant \|f\|_1 \cdot \|g\|_1 .$$

You only need to prove the estimate. Do not worry about measurability of $f * g$.

**Exercise 5.** **(15)** Let $(f_n)_{n=1}^\infty$ be a sequence of functions in $L^2([0, 1], m)$. Suppose that $\|f_n\|_2 \leqslant 17$ for every $n \in \mathbb{N}$. Let $f$ be an $m$-measurable function on $[0, 1]$ such that $f_n(x) \to f(x)$ for $m$-a.e. $x \in [0, 1]$.

**(a)** **(5)** Prove that $f \in L^2([0, 1], m)$ and $\|f\|_2 \leqslant 17$.

**(b)** **(10)** Prove that for every $g \in L^2([0, 1], m)$, we have

$$\lim_{n\to\infty} \int_{[0,1]} f_n(x) g(x) \,\mathrm{d}m(x) = \int_{[0,1]} f(x) g(x) \,\mathrm{d}m(x).$$

Hint for the second part: Egorov.

**Exercise 6.** **(10, 5 each)** Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $1 \leqslant p < \infty$.

**(a)** Give the definition of a bounded linear functional $H : L^p(X, \mu) \to \mathbb{R}$.

**(b)** State the Riesz representation theorem for bounded linear functionals on $L^p(X, \mu)$.