# Exam 3: Math 6320 Fall 2019
## Professor William Ott

**Exercise 1.** **(10)** Let $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ be an $(m \otimes m)$-measurable function such that for every $y \in [0,1]$, we have

$$\int_{\mathbb{R}} f^2(x,y)\,\mathrm{d}m(x) \leqslant 1.$$

Prove that there exists a sequence $x_n \to \infty$ such that

$$\int_{[0,1]} f(x_n, y)\,\mathrm{d}m(y) \to 0.$$

**Exercise 2.** **(15)** Let $1 \leqslant p < \infty$ and suppose $f : \mathbb{R} \to \mathbb{R}$ is an element of $L^p(m)$.

**(a)** **(5)** For every $\alpha > 0$, prove that the set

$$E_\alpha(f) = \{x \in \mathbb{R} : |f(x)| > \alpha\}$$

has finite Lebesgue measure.

**(b)** **(10)** Let $1 \leqslant q \leqslant 2$. Prove that every $g : \mathbb{R} \to \mathbb{R}$ in $L^q(m)$ can be decomposed as $g = g_1 + g_2$, where $g_1 \in L^1(m)$ and $g_2 \in L^2(m)$. Hint: Use part (a) and select a useful value of $\alpha$.

**Exercise 3.** **(10)** Let $1 \leqslant p \leqslant \infty$ and suppose $f : [0,\infty) \to \mathbb{R}$ is an element of $L^p(m)$. Prove that

$$\lim_{n\to\infty} \int_0^\infty f(x)e^{-nx}\,\mathrm{d}m(x) = 0.$$

**Exercise 4.** **(25)** This problem introduces you to some elements of dynamical systems and ergodic theory. Let $X$ be a compact metric space and let $T : X \to X$ be a continuous map. Let $\mathcal{M}(X)$ denote the set of Borel probability measures on $X$. A measure $\mu \in \mathcal{M}(X)$ is said to be $T$-invariant if $\mu(T^{-1}(E)) = \mu(E)$ for every Borel set $E$. Let $\mathcal{M}(X,T)$ denote the set of $T$-invariant Borel probability measures on $X$. We will prove next semester that $\mathcal{M}(X,T) \neq \emptyset$. This exercise establishes two important structural properties of $\mathcal{M}(X,T)$.

**(a)** **(5)** Prove that if $\mu$ and $\nu$ are elements of $\mathcal{M}(X,T)$ and if $p \in [0,1]$, then $p\mu + (1-p)\nu \in \mathcal{M}(X,T)$. (This shows that $\mathcal{M}(X,T)$ is a convex set.)

**(b)** **(10)** The ergodic measures in $\mathcal{M}(X,T)$ are of particular interest. A measure $\mu \in \mathcal{M}(X,T)$ is said to be ergodic for $T$ if whenever a Borel set $A$ satisfies $T^{-1}(A) = A$, then $\mu(A) = 0$ or $\mu(A) = 1$.

Suppose that $\mu_1$ and $\mu$ are members of $\mathcal{M}(X,T)$ such that $\mu_1 \ll \mu$ and $\mu$ is ergodic for $T$. Prove that $\mu_1 = \mu$. Hint: Let $f$ denote the Radon-Nikodym derivative $\mathrm{d}\mu_1/\mathrm{d}\mu$. Define $E = \{x \in X : f(x) < 1\}$. Prove that

$$\int_{E \setminus T^{-1}E} f\,\mathrm{d}\mu = \int_{T^{-1}E \setminus E} f\,\mathrm{d}\mu.$$

Argue that

$$\mu((T^{-1}E \setminus E) \cup (E \setminus T^{-1}E)) = 0.$$

This implies that $\mu(E) = 0$ or $\mu(E) = 1$ since $\mu$ is ergodic (you do not need to prove this implication). Show that we must have $\mu(E) = 0$. A similar argument (that you do not need to give) shows that $\mu(\{x \in X : f(x) > 1\}) = 0$ as well. Conclude that $f(x) = 1$ for $\mu$-a.e. $x \in X$ and therefore $\mu_1 = \mu$.

**(c)** **(10)** An element of $\mathcal{M}(X,T)$ is said to be an extreme point of $\mathcal{M}(X,T)$ if it cannot be written as a non-trivial convex combination of elements of $\mathcal{M}(X,T)$. Let $\mu \in \mathcal{M}(X,T)$. Prove that $\mu$ is ergodic for $T$ if and only if $\mu$ is an extreme point of $\mathcal{M}(X,T)$. Hint: For the forward implication, use part (b). For the reverse implication, argue by contraposition.