Math 6320 Fall 2019
Exam 1 Solutions

|E1b| Let $a \in \mathbb{R}$. Then for $x \in X$, $g(x) > a$ iff $f_n(x) > a$ for some $n \in \mathbb{N}$. Consequently,

$$g^{-1}((a,\infty)) = \bigcup_{n\in\mathbb{N}} f_n^{-1}((a,\infty)).$$

Now $f_n^{-1}((a,\infty)) \in \hat{M}$ for each $n \in \mathbb{N}$ since f_n is meas., so the countable union of these sets belongs to $\hat{M}$ since $\hat{M}$ is a σ-algebra.

|E2b| Define $F_1 = E_1$ and $F_{n+1} = E_{n+1} \setminus E_n$ for $n \geq 1$. Then $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$, so we have

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} F_i\right) \\ &= \sum_{i=1}^{\infty} \mu(F_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i) \\ &= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} F_i\right) \\ &= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} E_i\right). \quad \square\end{aligned}$$

|E4| First observe that for each $x \in [0,1]$, $\sqrt[n]{f(x)}$ converges to

$$g(x) = \begin{cases} 1, & \text{if } f(x) > 0; \\ 0, & \text{if } f(x) = 0. \end{cases}$$

Next observe that $\sqrt[n]{f(x)} \leq \max(1, f(x))$ for all $x \in X$ and $n \in \mathbb{N}$. The function $\max(1, f(x))$ is integrable, so LDCT implies

$$\begin{aligned}\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}\, dm(x) &= \int_0^1 g(x)\, dm(x) \\ &= m(\{x \in [0,1] : f(x) > 0\}). \quad \square\end{aligned}$$

5b By the monotone convergence theorem, we have

$$\sum_{n=1}^{\infty} \int_X |f_n(y) - f(y)| \, d\mu(y) = \int_X \underbrace{\left[\sum_{n=1}^{\infty} |f_n(y) - f(y)| \right]}_{g(y)} d\mu(y) < \infty.$$

Since g is integrable, g is finite almost everywhere. If $y \in X$ is such that $g(y) < \infty$, then $|f_n(y) - f(y)| \to 0$ as $n \to \infty$, since the terms of a convergent series must tend to zero. □

6c False. Counterexample:

$\chi_{[0,\frac{1}{2}]}, \chi_{[\frac{1}{2},1]}, \chi_{[0,1/4]}, \chi_{[1/4,1/2]}, \chi_{[\frac{1}{2},3/4]}, \chi_{[3/4,1]}, \ldots$

6d Let $\varepsilon > 0$. Define $A_n = \{x \in X : |f_n(x) - f(x)| > \varepsilon\}$. We work with the sequence (χ_{A_n}). By hypothesis, $\chi_{A_n}(y) \to 0$ for a.e. $y \in X$. Since μ is a finite measure, the constant function $\mathbb{1}$ is integrable. Since $\chi_{A_n} \leq \mathbb{1}$ $\forall n$, LDCT gives

$$0 = \lim_{n\to\infty} \int_X \chi_{A_n}(y) \, d\mu(y) = \lim_{n\to\infty} \mu(A_n). \quad \square$$