Math 6320 Fall 2019
Exam 2 Solutions

$\boxed{E1}$ (a) Let $\hat{C}_0$ denote the collection of finite unions of pairwise disjoint measurable rectangles. Then $\hat{M} \otimes \hat{A} = \sigma(\hat{C}_0)$, the σ-algebra generated by $\hat{C}_0$.

(b) Suppose $f : X \times Y \to \mathbb{R}$ is measurable with respect to $\hat{M} \otimes \hat{A}$. Suppose μ and ν are σ-finite measures. Then if either

(1) $f \geq 0$, or

(2) $\displaystyle\int_{X\times Y} |f(x,y)|\, d(\mu\times\nu)(x,y) < \infty$,

we have

$$\int_{X\times Y} f\, d(\mu\times\nu) = \int_Y \left[\int_X f(x,y)\, d\mu(x) \right] d\nu(y)$$
$$= \int_X \left[\int_Y f(x,y)\, d\nu(y) \right] d\mu(x) .$$

$\boxed{E2}$ This is an application of Fubini–Tonelli. We have

$$\int_{\mathbb{R}} |f(x)|\, dm(x)$$
$$= \int_{\mathbb{R}} \left[\int_{[0,\infty)} \mathbb{1}_{\{t : 0 \leq t \leq |f(x)|\}}\, dm(t) \right] dm(x)$$
$$\overset{(F-T)}{=} \int_{[0,\infty)} \int_{\mathbb{R}} \mathbb{1}_{\{t : 0 \leq t \leq |f(x)|\}}\, dm(x)\, dm(t)$$
$$= \int_{[0,\infty)} m\{x \in \mathbb{R} : |f(x)| \geq t\}\, dm(t) .$$

$\boxed{E3}$ (a) Apply Hölder. If $f \in L^q$, then $|f|^q = |f|^{p\cdot(q/p)}$ is integrable. Observe that

$$1 = \frac{1}{(q/p)} + \frac{1}{r}$$

for some r.

We have

$$\int_X |f|^p d\mu = \int_X |f|^p \cdot \mathbb{1} \, d\mu$$
$$\leq \left[\int_X (|f|^p)^{q/p} d\mu \right]^{p/q} \left[\int_X (\mathbb{1})^r d\mu \right]^{1/r}$$
$$= \left[\int_X |f|^q d\mu \right]^{p/q} \left[\mu(X) \right]^{1/r}.$$

This shows that $f \in L^q \Rightarrow f \in L^p$.

(b) We give a simple example with $X = \mathbb{R}$ and $\mu = m$. The function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{1}{x} \mathbb{1}_{[1,\infty)}(x)$ is in $L^2(\mathbb{R}, m)$, but not $L^1(\mathbb{R}, m)$.

$\boxed{E4}$ We have

$$\int_{\mathbb{R}} |((f)*(g))(x)| \, dm(x)$$
$$= \int_{\mathbb{R}} \left| \left[\int_{\mathbb{R}} f(x-y) g(y) \, dm(y) \right] \right| dm(x)$$
$$\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y) g(y)| \, dm(y) \right] dm(x)$$
$$\overset{(F-T)}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y) g(y)| \, dm(x) \, dm(y)$$
$$= \int_{\mathbb{R}} \|f\|_{L^1} |g(y)| \, dm(y) \quad \text{(since } m \text{ is translation-invariant)}$$
$$= \|f\|_{L^1} \|g\|_{L^1}.$$

$\boxed{E5}$ (a) Using Fatou, $\int_{[0,1]} f^2 \, dm \leq \liminf_{n \to \infty} \int_{[0,1]} f_n^2 \, dm \leq 17^2$. Hence $\|f\|_{L^2} \leq 17$.

$\boxed{E5b}$ We use a large set - small set decomposition. Suppose A is a measurable set on which $f_n \to f$ uniformly. We have

$$\int_{[0,1]} |(f_n - f)g|\,dm = \int_A |(f_n-f)g|\,dm + \int_{A^c} |(f_n-f)g|\,dm$$
$$\leq \Big[\sup_{x\in A} |f_n(x) - f(x)|\Big]\cdot \|g\|_{L^1} + \|f_n - f\|_{L^2}\|g\chi_{A^c}\|_{L^2}$$
$$(*) \quad \leq \Big[\sup_{x\in A} |f_n(x) - f(x)|\Big]\cdot \|g\|_{L^1} + \Big[\|f_n\|_{L^2} + \|f\|_{L^2}\Big]\cdot \|g\chi_{A^c}\|_{L^2}.$$

Here we have used Hölder and Minkowski for the second term. Now let $\varepsilon > 0$. Using Egorov and the fact that $g \in L^2$, we can find A such that $f_n \to f$ uniformly on A and $\|g\chi_{A^c}\|_{L^2} < \varepsilon/(4\cdot 17)$. Next, $\|g\|_{L^1} < \infty$, since $L^2 \subset L^1$ because the measure space is finite. Choose $N \in \mathbb{N}$ such that $n \geq N \Rightarrow$
$\sup_{x\in A} |f_n(x) - f(x)| < \varepsilon/(2\cdot\|g\|_{L^1})$. Putting all of this into $(*)$, for $n \geq N$ we have $\int_{[0,1]} |(f_n - f)g|\,dm < \varepsilon$. □

$\boxed{E6}$ (a) H is a bounded linear functional if H is a linear map and $\|H\| = \sup_{\substack{f\in L^p(X,\mu)\\ \|f\|_p = 1}} |Hf| < \infty$.

(b) If μ is σ-finite and H is a bounded linear functional, $\exists\ g \in L^q$ such that $H(f) = \int_X fg\,d\mu$ and $\|g\|_q = \|H\|$. Here $p^{-1} + q^{-1} = 1$.