Math 6320 Fall 2019
Exam 3 Solutions

E1 Applying Fubini–Tonelli, we have

$$\int_{\mathbb{R}} \int_{[0,1]} f^2(x,y)\,dy\,dx$$
$$= \int_{[0,1]} \int_{\mathbb{R}} f^2(x,y)\,dx\,dy \leq \int_{[0,1]} 1\,dy = 1.$$

Let $g(x) = \int_{[0,1]} f^2(x,y)\,dy$. By the above, g is integrable on $\mathbb{R}$. There must exist a seq. $x_n \to \infty$ such that $g_n = g(x_n) \to 0$. If not, $\exists$ $\varepsilon > 0$ and $L > 0$ such that $g(x) \geq \varepsilon$ $\forall x \geq L$, contradicting the integrability of g.

At this point, we have

$$\int_{[0,1]} f^2(x_n,y)\,dy \to 0.$$

Using Proposition 6.12 of Folland,

$$\int_{[0,1]} |f(x_n,y)|\,dy \leq \left[\int_{[0,1]} f^2(x_n,y)\,dy \right]^{1/2}.$$

Finally, observe that $\left| \int_{[0,1]} f(x_n,y)\,dy \right| \leq \int_{[0,1]} |f(x_n,y)|\,dy$. The proof is complete.

E2 (a) The Chebyshev inequality gives an easy estimate of $m(E_\alpha(f))$. We have

$$\|f\|_p^p = \int_{\mathbb{R}} |f|^p\,dm$$
$$\geq \int_{E_\alpha(f)} |f|^p\,dm$$
$$\geq \alpha^p\, m(E_\alpha(f)).$$

Consequently, $m(E_\alpha(f)) \leq \dfrac{\|f\|_p^p}{\alpha^p}$.

$\boxed{E2b}$ Note that the problem is trivial if $q=1$ or $q=2$. If $q \in (1,2)$, set $\alpha = 1$ and write

$$g(x) = \underbrace{g(x)\chi_{E_1(g)}(x)}_{h(x)} + \underbrace{g(x)\chi_{\mathbb{R}\setminus E_1(g)}(x)}_{k(x)}.$$

We have $h \in L^1(\mathbb{R}, m)$, since

$$\begin{aligned}\|h\|_1 &= \int_{\mathbb{R}} |h(x)|\,dm(x)\\ &= \int_{E_1(g)} |g(x)|\,dm(x)\\ &\leq \int_{E_1(g)} |g(x)|^q\,dm(x)\\ &\leq \int_{\mathbb{R}} |g(x)|^q\,dm(x) = \|g\|_q^q < \infty.\end{aligned}$$

We have $k \in L^2(\mathbb{R}, m)$, since

$$\begin{aligned}\|k\|_2^2 &= \int_{\mathbb{R}} k^2(x)\,dm(x)\\ &= \int_{\mathbb{R}\setminus E_1(g)} g^2(x)\,dm(x)\\ &\leq \int_{\mathbb{R}\setminus E_1(g)} |g(x)|^q\,dm(x)\\ &\leq \int_{\mathbb{R}} |g(x)|^q\,dm(x) = \|g\|_q^q < \infty.\end{aligned}$$

The proof is complete.

$\boxed{E3}$ We consider two cases.

- $p=1$

Observe that $f(x)e^{-nx} \xrightarrow{n\to\infty} g(x) = \begin{cases} 0, & x>0; \\ f(x), & x=0. \end{cases}$

Next observe that $|f(x)e^{-nx}| \leq |f(x)|\ \forall n$, and $|f|$ is integrable ($p=1$), so Lebesgue DCT implies

$$\int_{[0,\infty)} f(x)e^{-nx}\,dm(x) \xrightarrow[n]{} \int_{[0,\infty)} g(x)\,dm(x) = 0.$$

$p > 1$

Let q satisfy $p^{-1} + q^{-1} = 1$. Applying Hölder, we have

$$\left| \int_{[0,\infty)} f(x) e^{-nx} \, dm(x) \right| \leq \int_{[0,\infty)} |f(x)| e^{-nx} \, dm(x)$$

$$\leq \|f\|_p \|e^{-nx}\|_q .$$

Since $p > 1$, $q < \infty$, so

$$\|e^{-nx}\|_q^q = \int_{[0,\infty)} e^{-nqx} \, dm(x)$$

$$= \left(-\tfrac{1}{nq}\right) e^{-nqx} \Big|_0^\infty$$

$$= 1/(nq) .$$

Clearly $1/(nq) \to 0$ as $n \to \infty$. $\square$

$\boxed{E4}$ (a) Let $\mu, \nu \in M(X,T)$ and let $p \in [0,1]$. Let E be a Borel set. We have

$$\begin{aligned}(p\mu + (1-p)\nu)(T^{-1}(E)) &= p\mu(T^{-1}(E)) + (1-p)\nu(T^{-1}(E)) \\ &= p\mu(E) + (1-p)\nu(E) \\ &= (p\mu + (1-p)\nu)(E).\end{aligned}$$

Therefore $p\mu + (1-p)\nu \in M(X,T)$.

(b) Suppose $\mu_1, \mu \in M(X,T)$, $\mu_1 \ll \mu$, and μ is ergodic. We prove $\mu_1 = \mu$. To do so, let f denote the Radon–Nikodym derivative $d\mu_1/d\mu$. We will show that $f(x) = 1$ for μ-a.e. $x \in X$.

Define $E = \{x \in X : f(x) < 1\}$. We have

$$\begin{aligned}\int_{E\setminus T^{-1}E} f\,d\mu + \int_{E\cap T^{-1}E} f\,d\mu &= \mu_1(E) \\ &= \mu_1(T^{-1}E) \\ &= \int_{T^{-1}E\setminus E} f\,d\mu + \int_{T^{-1}E\cap E} f\,d\mu,\end{aligned}$$

giving

$$\int_{E\setminus T^{-1}E} f\,d\mu = \int_{T^{-1}E\setminus E} f\,d\mu.$$

Next,

$$\begin{aligned}\mu(E\setminus T^{-1}E) &= \mu(E) - \mu(E\cap T^{-1}E) \\ &= \mu(T^{-1}E) - \mu(E\cap T^{-1}E) \\ &= \mu(T^{-1}E\setminus E),\end{aligned}$$

and $f < 1$ on $E\setminus T^{-1}E$ while $f \geq 1$ on $T^{-1}E\setminus E$, so we must have $\mu(E\setminus T^{-1}E) = \mu(T^{-1}E\setminus E) = 0$. Since μ is ergodic, $\mu(E) \in \{0,1\}$. But $\mu(E) = 1$ would force $\mu_1(X) = \int_E f\,d\mu < 1$, contradicting $\mu_1(X) = 1$. Thus $\mu(E) = 0$.

Similarly, $\mu\{x \in X : f(x) > 1\} = 0$. We conclude that $f(x) = 1$ for μ-a.e. $x \in X$, so $\mu_1 = \mu$.

(c) ($\Rightarrow$) Suppose μ is ergodic. Suppose $\mu = p\nu_1 + (1-p)\nu_2$, where $p \in [0,1]$ and $\nu_1, \nu_2 \in M(X,T)$. Then $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, so $\nu_1 = \nu_2 = \mu$ by (b). The components in the convex combination are therefore trivial, so μ is an extreme point.

($\Leftarrow$) Suppose μ is not ergodic. Then $\exists$ a Borel set B such that $T^{-1}B = B$ and $\mu(B) \in (0,1)$. Define T-invariant measures in $M(X,T)$ by

$$\nu_1(A) = \frac{\mu(A \cap B)}{\mu(B)}, \quad \nu_2(A) = \frac{\mu(A \cap B^c)}{\mu(B^c)}.$$

We then have $\mu = \mu(B)\nu_1 + \mu(B^c)\nu_2$, so μ is not an extreme point.