Exam 2: Math 1432 Spring 2018
Professor William Ott

Exercise 1. **(15)** Let $\mathcal{R}$ denote the region bounded by the curves $y = 0$, $y = e^x$, $x = 0$, and $x = 1$.

(a) **(5)** Find the area of $\mathcal{R}$.

(b) **(10)** Find the center of mass of $\mathcal{R}$.

Exercise 2. **(10)** Consider the parametric curve defined for all real t by

$$\begin{cases} x(t) = t^2 - 2t, \\ y(t) = t^3 - 3t. \end{cases}$$

Find the point(s) on the curve at which a tangent line exists and is horizontal.

Exercise 3. **(10, 5 each)** Consider the parametric curve defined by

$$\begin{cases} x(t) = \dfrac{1}{2}t^2 \\ y(t) = \ln(t) \end{cases}$$

for $t \geqslant \frac{1}{3}$.

(a) For $z > \frac{1}{3}$, let $s(z)$ be the arc length of the parametric curve from $t = \frac{1}{3}$ to $t = z$. Set up, but do not evaluate, a $\mathrm{d}t$ integral for this length.

(b) At what value of z in $(\frac{1}{3}, \infty)$ does $s(z)$ have an inflection point?

Exercise 4. **(10, 5 each)** Suppose that we rotate the infinite curve $y = e^{-x}$ $(x \geqslant 0)$ about the x-axis.

(a) Set up, but do not evaluate, an integral for the surface area of the resulting surface of revolution.

(b) Does the surface area integral you set up converge or diverge? Justify your answer.

Exercise 5. **(10)** Find the slope of the tangent line to the graph of the polar curve $r = \frac{1}{\theta}$ at the point on the graph specified by $\theta = \pi$.

Exercise 6. **(10)** Find the area inside the curve $r = 4\sin(\theta)$ and outside the curve $r = 2$.

Exercise 7. **(10)** Let $0 < a < b$. Suppose that f and g are continuous functions on $[a, b]$ such that $f(x) > g(x) > 0$ for every x in $[a, b]$. Let $\mathcal{R}$ denote the region bounded by the graphs of f and g, as well as the lines $x = a$ and $x = b$. Prove that if $\mathcal{R}$ is rotated about the y-axis, the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

Exercise 8. **(Bonus 10)** The curvature κ at a point P of a curve is defined as

$$\kappa = \left| \frac{\mathrm{d}\phi}{\mathrm{d}s} \right|,$$

where ϕ is the angle of inclination of the tangent line at P. That is, the curvature is the absolute value of the rate of change of ϕ with respect to arc length. For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}.$$

Here overdot denotes the first derivative with respect to time t and double overdot denotes the second time derivative. Hints:

$$\phi = \tan^{-1}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right), \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right).$$