ANALYSIS QUALIFYING EXAMINATION

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Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. You may use any given hint without proving it. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

Problem 1.

Let p be a non-constant real polynomial on $\mathbb{R}$. Show that

$$\lim_{n\to+\infty}\int_0^1 \exp(2\pi i n p(x))\,dx = 0\ .$$

Problem 2.

Prove that the following indefinite integral converges and evaluate it explicitly:

$$\int_0^\infty \frac{\log x}{x^2+4}dx\ .$$

Problem 3.

Let $(\mathbb{R}, \mathcal{M}, m)$ denote the Lebesgue measure space. Let $\{f_n : n = 1, 2, \ldots\}$ be a decreasing sequence of measurable functions (i.e., for any $x \in \mathbb{R}$ and $n \geq 1$, $f_n(x) \geq f_{n+1}(x)$) such that $f_1 \in L^1(\mathbb{R})$ and $f_n \to 0$, $m - a.e.$, as $n \to +\infty$. Show that for every $\varepsilon > 0$ there exists a Lebesgue measurable set A with $m(A) < \varepsilon$ such that f_n converges uniformly to 0 on $A^\sim$ (the complement of A).

Problem 4.

Let $f(z)$ be an analytic function on $\mathbb{C} \setminus \{0\}$, and suppose that

$$|f(z)| \leq \sqrt{|z|} + \frac{1}{\sqrt{|z|}}\ ,$$

for all $z \in \mathbb{C} \setminus \{0\}$. Prove that $f(z)$ must be a constant function.

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Problem 5.

Let $f \in L^1(\mathbb{R})$ and m denote the Lebesgue measure. Show that

$$\lim_{t \to +\infty} t \cdot m(\{x : |f(x)| \geq t\}) = 0 \ .$$

Problem 6.

Prove that the series below determines a meromorphic function on $\mathbb{C}$. Find all of its poles and their orders.

$$\sum_{n=1}^{\infty} \frac{1}{z^2 + n^2} \ .$$