ANALYSIS QUALIFYING EXAMINATION

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MATHEMATICS DEPARTMENT
UNIVERSITY OF MARYLAND

Unless otherwise stated, you may appeal to a "well known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified.

Notation: Throughout the exam, D denotes the unit disk in the complex plane and m denotes Lebesgue measure on the real line.

Problem 1.

Let $\{f_n, n = 1, \ldots\}$ be a sequence of non-negative Lebesgue measurable functions on a bounded interval $[a, b]$, $a < b$, such that $f_n \to 0$ m-*a.e.* Use Egorov's theorem to show that there exists an infinite subsequence $\{f_{n_k} : k = 1, \ldots\}$ of $\{f_n : n = 1, \ldots\}$, such that

$$\sum_{k=1}^{\infty} f_{n_k} < \infty \quad m\text{-}a.e.$$

Problem 2.

Suppose $f(z)$ is an analytic function on a connected open neighborhood of the closed unit disk and $|f(z)| = 1$ whenever $|z| = 1$. Find a formula for $f(z)$ (Hint: first consider the case where $f(z)$ has no zeros in the unit disk).

Problem 3.

Let f and $\{f_n, n = 1, \ldots\}$ be Lebesgue measurable functions on $\mathbb{R}$ such that $\{f, f_n, n = 1, \ldots\} \subset L_m^{2020}(\mathbb{R})$. Assume that $\lim_{n\to\infty} f_n = f$ m-*a.e.* Assume also that $\lim_{n\to\infty} \|f_n\|_{2020} = \|f\|_{2020}$. Prove that $\lim_{n\to\infty} \|f_n - f\|_{2020} = 0$. You may use here the fact that for any $p \geq 1$, we have $|a + b|^p \leq 2^p(|a|^p + |b|^p)$, provided that you can prove it.

Problem 4.

Let $f : D \to \mathbb{C}$ be an analytic function on the disk with $f(0) = 1$, and suppose that

$$f(D) \subset \{w \mid \operatorname{Re} w > 0\} \ .$$

Prove that

$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}$$

for all $z \in D$.

Problem 5.

Consider the family of functions on $[a, b]$ which satisfy the property that for every $\epsilon > 0$ there exists $\delta > 0$ such that for every family of (possibly overlapping) intervals $(a_k, b_k) \subset [a, b]$, $k = 1, \ldots, n$, with $\sum_{k=1}^{n}(b_k - a_k) < \delta$, we have $\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$. Prove that this family of functions coincides with the family of Lipschitz continuous functions. (Recall that a function $f : [a, b] \to \mathbb{R}$ is Lipschitz continuous if there exists $K > 0$ for which $|f(x) - f(y)| < K|x - y|$ for all $x, y \in [a, b]$.)

Problem 6.

Fix a complex number $0 < |\lambda| < 1$. For $n \geq 1$, let

$$f(z) = (z - 1)^n e^z + \lambda(z + 1)^n \ .$$

Show that $f(z)$ has exactly n simple zeros with $\operatorname{Re} z > 0$.