Analysis PhD Qualifying Examination: January 2016

Instructions. Each exercise is worth 10 total points. Please solve 6 of the 10 exercises, subject to the constraint that you must solve at least two of the final four exercises. In the *Graded exercises* area, please clearly list the 6 exercises you wish to have graded. Whenever you provide a counterexample, you must prove that your counterexample works.

Notation and conventions. $\mathcal{F}[\cdot]$ denotes the Fourier transform. Let m denote one-dimensional Lebesgue measure. All functions are real-valued unless explicitly stated otherwise. Euclidean spaces and subsets thereof are equipped with Lebesgue measure (unless specifically stated otherwise).

Graded exercises:

Exercise 1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Prove that if $\mu(X) < \infty$ and if $1 \leqslant p < q < \infty$, then $L^q(\mu) \subset L^p(\mu)$.

(b) Is the statement in (a) true if $\mu(X) = \infty$? If yes, prove it. If no, give a counterexample.

Exercise 2. Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{x^n \cos(x/n)}{(1+x^n)e^x} \, \mathrm{d}x$$

exists and compute it.

Exercise 3. Let $f \in L^1((0,1))$. Define g on $(0,1)$ by

$$g(x) = \int_x^1 \frac{f(t)}{t} \, \mathrm{d}t.$$

Prove that $g \in L^1((0,1))$.

Exercise 4. Fix $1 < p < \infty$ and let q satisfy $1/p + 1/q = 1$. Let $(f_n)_{n=1}^\infty$ be a sequence in $L^p([0,1])$ for which there exists $K > 0$ such that $\|f_n\|_p \leqslant K$ for every $n \in \mathbb{N}$. Suppose that there exists a Lebesgue measurable function f on $[0,1]$ such that $f_n(x) \to f(x)$ for m-a.e. $x \in [0,1]$.

(a) Prove that $f \in L^p([0,1])$ and $\|f\|_p \leqslant K$.

(b) Prove that for every $g \in L^q([0,1])$, we have

$$\lim_{n\to\infty} \int_0^1 f_n(x) g(x) \, \mathrm{d}x = \int_0^1 f(x) g(x) \, \mathrm{d}x.$$

(c) Is the statement in part (b) true if $p = 1$ and $q = \infty$? If yes, prove it. If no, give a counterexample.

Exercise 5. (On modes of convergence) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n=1}^\infty$ be a sequence of μ-integrable functions and suppose f is μ-integrable as well.

(a) Prove that if $f_n \to f$ in the $L^1(\mu)$ sense, then $f_n \to f$ in measure.

(b) If $\mu(X) < \infty$ and if $f_n \to f$ in measure, does it follow that $f_n \to f$ in the $L^1(\mu)$ sense? Either prove this or give a counterexample.

Exercise 6. (On a property of Lebesgue integrable functions) Let $f \in L^1(\mathbb{R})$.

(a) Fix $\alpha > 0$. For $n \in \mathbb{N}$, define f_n by $f_n(x) = f(nx)/n^\alpha$. Show that

$$\|f_n\|_1 = \int_{\mathbb{R}} \frac{|f(nx)|}{n^\alpha} \, \mathrm{d}x = \int_{\mathbb{R}} \frac{|f(z)|}{n^{1+\alpha}} \, \mathrm{d}z = \frac{\|f\|_1}{n^{1+\alpha}}.$$

(b) Use (a) to show that $f_n(x) \to 0$ as $n \to \infty$ for m-a.e. $x \in \mathbb{R}$.

Exercise 7. Let $F \subset \mathbb{R}$ be a closed set of positive measure. For $x \in \mathbb{R}$, define the distance from x to F by

$$d(x, F) = \inf_{z \in F} d(x, z).$$

Prove that for Lebesgue almost every $y \in F$, we have

$$\lim_{x \to y} \frac{d(x, F)}{|x - y|} = 0.$$

Hint: Consider Lebesgue density points of F.

Exercise 8. (On absolute continuity)

- **(a)** Let $a < b$ be real numbers. Give the definition of an absolutely continuous function $f : [a, b] \to \mathbb{R}$.
- **(b)** Suppose $f : [a, b] \to \mathbb{R}$ is absolutely continuous. Prove that if A is a Lebesgue measurable subset of $[a, b]$ with $m(A) = 0$, then $m(f(A)) = 0$.
- **(c)** If E is a Lebesgue measurable subset of $\mathbb{R}$ with $m(E) = 0$, does it follow that

$$\{e^x : x \in E\}$$

has Lebesgue measure zero? Either prove this or give a counterexample.

Exercise 9. (On the Fourier transform) Let $f : \mathbb{R} \to \mathbb{C}$ be Lebesgue integrable. Recall that the Fourier transform of f is defined by

$$\hat{f}(\gamma) = \int_{\mathbb{R}} f(t) e^{-2\pi i \gamma t} \, \mathrm{d}t.$$

- **(a)** Prove that $\hat{f}$ is uniformly continuous on $\mathbb{R}$.
- **(b)** Prove that

$$\lim_{\gamma \to \infty} \hat{f}(\gamma) = 0.$$

Hint: First show this for the characteristic function of an interval of finite length. To complete the proof, make a density argument.

Exercise 10. (On weak convergence) Let (f_n) be a sequence of functions in $L^2([0, 1])$ that converges weakly to $f \in L^2([0, 1])$, meaning that

$$\lim_{n \to \infty} \int_0^1 f_n g \, \mathrm{d}m = \int_0^1 f g \, \mathrm{d}m$$

for every $g \in L^2([0, 1])$. Prove that there exists $K > 0$ such that $\|f_n\|_{L^2([0,1])} \leqslant K < \infty$ for every $n \in \mathbb{N}$. Hint: Uniform boundedness principle.