Assignment 3: Math 6321 Spring 2020

Exercise 1. (5.5.55) Let $\mathcal{H}$ be a Hilbert space.

(a) Prove that for every $x, y \in \mathcal{H}$, we have

$$\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2).$$

This is known as the polarization identity.

(b) Let $\mathcal{H}'$ be another Hilbert space. Prove that a linear map A from $\mathcal{H}$ to $\mathcal{H}'$ is unitary iff A is isometric and surjective.

Exercise 2. (5.5.58) Let $\mathcal{M}$ be a closed subspace of the Hilbert space $\mathcal{H}$. For $x \in \mathcal{H}$, let Px be the element of $\mathcal{M}$ such that $x - Px \in \mathcal{M}^\perp$.

(a) Prove that $P \in L(\mathcal{H}, \mathcal{H})$, $P^* = P$, $P^2 = P$, $\mathcal{R}(P) = \mathcal{M}$, and $\mathcal{N}(P) = \mathcal{M}^\perp$. The operator P is the orthogonal projection onto $\mathcal{M}$.

(b) Conversely, suppose that $Q \in L(\mathcal{H}, \mathcal{H})$ satisfies $Q^2 = Q^* = Q$. Prove that $\mathcal{R}(Q)$ is closed and Q is the orthogonal projection onto $\mathcal{R}(Q)$.

(c) Show that if $\{u_\alpha\}$ is an orthonormal basis for $\mathcal{M}$, then the orthogonal projection P onto $\mathcal{M}$ is given by $Px = \sum_\alpha \langle x, u_\alpha\rangle u_\alpha$.

Exercise 3. (5.5.63) Let $\mathcal{H}$ be an infinite-dimensional Hilbert space.

(a) Prove that every orthonormal sequence in $\mathcal{H}$ converges weakly to 0.

(b) Show that the unit sphere $S = \{x \in \mathcal{H} : \|x\| = 1\}$ is weakly dense in the closed unit ball $B = \{x \in \mathcal{H} : \|x\| \leqslant 1\}$.

Exercise 4. (5.5.67) Let U be a unitary operator on the Hilbert space $\mathcal{H}$. Let $\mathcal{M} = \{x \in \mathcal{H} : Ux = x\}$ and let P be the orthogonal projection onto $\mathcal{M}$. Define

$$S_n = \frac{1}{n}\sum_{j=0}^{n-1} U^j.$$

Prove that $S_n \to P$ in the strong operator topology. This result is known as the mean ergodic theorem. You are free to use any result from Exercise 5.5.57 in your proof.

Exercise 5. (5 - Maryland August 2013) (In this problem, you may not appeal to the Birkhoff pointwise ergodic theorem.) Let m denote Lebesgue measure on $[0,1]$. Suppose $T : [0,1] \to [0,1]$ is a Lebesgue measurable function such that for every measurable set $E \subset [0,1]$, if $T^{-1}(E) = E$, then either $m(E) = 0$ or $m([0,1] \setminus E) = 0$.

(a) Suppose $f : [0,1] \to \mathbb{R}$ is a Lebesgue measurable function such that $f(x) = f(T(x))$ for all $x \in [0,1]$. Prove that there exists $c \in \mathbb{R}$ such that $\{x : f(x) = c\}$ has measure 1.

(b) Suppose also that for every Lebesgue measurable subset E of $[0,1]$ with $m(E) > 0$, there exists a subset F of $[0,1]$ with $m([0,1] \setminus F) = 0$ such that for every $x \in F$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbf{1}_E(T^k(x)) = m(E).$$

Given $f : [0,1] \to \mathbb{R}$ bounded and measurable, prove that the sequence $(f_n)_{n=1}^\infty$ defined by

$$f_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(T^k(x))$$

converges to $\int_0^1 f\,\mathrm{d}m$ in $L^1(m)$.