Math 2415: Some formulas, examples, and quiz problems

Part I.

1. "**No Calculators are allowed on the exam.**"

2. "**Don't miss too many quizzes.**"

3. "**Work on Part II and turn in your answers to your TA on June 5, 2024.**"

4. **Some formulas:**

   (a) $\cos^2 t + \sin^2 t = 1, \ \cos^2(2x) + \sin^2(2x) = 1, \ \cos^2(s^2) + \sin^2(s^2) = 1.$

   (b) $(x+y)^2 = x^2 + 2xy + y^2.$

   (c) $(x-y)^2 = x^2 - 2xy + y^2.$

   (d) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$

   (e) $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3.$

   Examples:

   i. $(x-3)^2 = x^2 - 6x + 9;$

   ii. $4t^2 + 4t + 1 = (2t)^2 + 2 \cdot 2t + 1 = (2t+1)^2.$

   iii. $t^2 + 4 + \dfrac{4}{t^2} = t^2 + 2t \cdot \dfrac{2}{t} + (\dfrac{2}{t})^2 = (t + \dfrac{2}{t})^2.$

   iv. $(x-2)^3 = x^3 - 6x^2 + 12x - 8;$

   v. How to complete the square?

   $x^2 + ax + b = x^2 + ax + (\frac{a}{2})^2 + b - (\frac{a}{2})^2 = (x + \frac{a}{2})^2 + b - \frac{a^2}{4}.$

   $x^2 + 6x + 1 = (x^2 + 6x + 3^2) + 1 - 3^2 = (x+3)^2 - 8.$

   $t^2 - 8t + 7 = (t^2 - 8t + 4^2) + 7 - 4^2 = (t-4)^2 - 9.$

   (f) $x^2 + (a+b)x + ab = (x-a)(x-b).$

   Examples: $x^2 + 6x + 8 = (x+2)(x+4), \ \ t^2 - 5t + 4 = (t-1)(t-4).$

   (g) $\sqrt{(x+y)^2} = |x+y|.$

   Examples: $\sqrt{t^2 + 4t + 4} = \sqrt{(t+2)^2} = |t+2|.$

$$\int_{-4}^{2} \sqrt{t^2+4t+4}\, dt = \int_{-4}^{2} \sqrt{(t+2)^2}\, dt = \int_{-4}^{2} |t+2|\, dt$$

$$= \int_{-4}^{-2} -(t+2)\, dt + \int_{-2}^{2} (t+2)\, dt = -(\frac{1}{2}t^2 + 2t)\Big|_{t=-4}^{-2} + (\frac{1}{2}t^2 + 2t)\Big|_{t=-2}^{2} = 10.$$

   Don't do the following since they are wrong.

   i. $\sqrt{2} + \sqrt{2} = \sqrt{4} = 2.$

   ii. $\sqrt{x^2 + y^2} = \sqrt{x^2} + \sqrt{y^2} = x + y.$

   iii. $\displaystyle\int_1^2 \sqrt{t^2+4t+4}\, dt = \int_1^2 (\sqrt{t^2} + \sqrt{4t} + \sqrt{4})\, dt.$

   iv. $\displaystyle\int_{-4}^{2} \sqrt{t^2+4t+4}\, dt = \int_{-4}^{2} \sqrt{(t+2)^2}\, dt = \int_{-4}^{2} (t+2)\, dt = (\frac{1}{2}t^2 + 2t)\Big|_{t=-4}^{2} = 6.$

5. **Domain and continuity of some basic functions**:

(a) A polynomial, e.g., $f(x) = 2x^5 + 4x - 10$, is defined and continuous for all real numbers.

(b) $f(x) = \cos x$ and $g(x) = \sin x$ are defined and continuous for all real numbers.

(c) $f(x) = e^x$ is defined and continuous for all real numbers.

(d) $f(x) = \ln x$ is defined and continuous for all positive real numbers (i.e., $x > 0$).

(e) $f(x) = \sqrt{x}$ is defined for all non-negative real numbers (i.e., $x \geq 0$) and continuous for all positive real numbers (i.e., $x > 0$).

(f) $f(x) = \dfrac{1}{x}$ is defined and continuous for all non-zero real numbers (i.e., $x \neq 0$).

6. **The sine and cosine of special angles**:

(a) $\sin 0 = 0,\ \sin\left(\frac{\pi}{6}\right) = \frac{1}{2},\ \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \sin\left(\frac{\pi}{2}\right) = 1,$
$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2},\ \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \sin\left(\frac{5\pi}{6}\right) = \frac{1}{2},\ \sin(\pi) = 0$

(b) $\cos 0 = 1,\ \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},\ \cos\left(\frac{\pi}{3}\right) = \frac{1}{2},\ \cos\left(\frac{\pi}{2}\right) = 0,$
$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2},\ \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2},\ \cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2},\ \cos(\pi) = -1$

7. **Some formulas for finding derivatives**:

(a) Some basic results:

i. Given $f(x) = c$ for some constant c, $f'(x) = 0$.

ii. Given $f(x) = x^n$ for some non-zero integer n, $f'(x) = nx^{n-1}$.

iii. For $f(x) = \cos x$, $f'(x) = -\sin x$.

iv. For $f(x) = \sin x$, $f'(x) = \cos x$.

v. For $f(x) = e^x$, $f'(x) = e^x$.

vi. For $f(x) = \ln x$, $f'(x) = 1/x$ for $x > 0$.

vii. For $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ for $x > 0$.

(b) Some basic rules for finding derivatives:

i. $(a\,f(x) + b\,g(x))' = a\,f'(x) + b\,g'(x)$:

Example: $\dfrac{d}{dx}\left(5x^3 + 4x - 2\cos x\right) = 15x^2 + 4 + 2\sin x.$

ii. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$:

Example: $\dfrac{d}{dx}((5x^3 + 4x)\cos x) = (15x^2 + 4)\cos x - (5x^3 + 4x)\sin x.$

iii. $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$:

Example: $\dfrac{d}{dx}\left(\dfrac{5x^3 + 4x}{\cos x}\right) = \dfrac{(15x^2 + 4)\cos x + (5x^3 + 4x)\sin x}{\cos^2 x}.$

iv. $(f(u(x)))' = f'(u(x))u'(x)$:

Example:

A. $\dfrac{d}{dx}\left((5x^3 + 4x)^3\right) = 3(15x^2 + 4)\,(5x^3 + 4x)^2.$

B. $\dfrac{d}{dx}\left(\cos(5x^3 + 4x)\right) = -(15x^2 + 4)\,\sin(5x^3 + 4x).$

C. $\dfrac{d}{dx}\left(e^{5x^3+4x}\right) = (15x^2 + 4)\,e^{5x^3+4x}.$

D. $\dfrac{d}{dx}\left(\ln(5x^3 + 4x)\right) = \dfrac{15x^2 + 4}{5x^3 + 4x}.$

E. $\dfrac{d}{dx}\left(\sqrt{5x^3 + 4x}\right) = \dfrac{15x^2 + 4}{2\sqrt{5x^3 + 4x}}.$

8. **Some integration formulas**:

(a) $\int dx = x + c.$

(b) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$ for $n \neq -1$.

(c) $\int \frac{1}{x}\,dx = \ln|x| + c.$

(d) $\int \sqrt{x}\,dx = \frac{2x\sqrt{x}}{3} + c.$

Examples:

i. $\int x^2\,dx = \frac{1}{3}x^3 + c$

ii. $\int x^5\,dx = \frac{1}{6}x^6 + c$

iii. $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$

(e) $\int \sin x\,dx = -\cos x + c.$

(f) $\int \cos x\,dx = \sin x + c.$

(g) $\int e^x\,dx = e^x + c.$

(h) $\int (a\ f(x) + b\ g(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx.$

(i) $\int f'(x)u'(x)\,dx = f(u(x)) + c.$ (u-substitution)

Examples:

i. $\int (4x^4 - 7x^2 - 5)\,dx = \frac{4}{5}x^5 - \frac{7}{3}x^3 - 5x + c.$

ii. $\int x\cos 3x^2\,dx = \frac{1}{6}\sin 3x^2 + c.$ $(u(x) = 3x^2)$

iii. $\int 2xe^{5x^2+2}\,dx = \frac{1}{5}e^{5x^2+2} + c.$

iv. $\int \frac{2x-1}{2x^2-2x+5}\,dx = \frac{1}{2}\ln|2x^2 - 2x + 5| + c.$

(j) $\int_a^b f'(x)\,dx = f(x)\Big|_{x=a}^{b} = f(b) - f(a).$

Examples:

i. $\int_0^1 (4x^4 - 7x^2 - 5)\,dx = \left(\frac{4}{5}x^5 - \frac{7}{3}x^3 - 5x\right)\Big|_{x=0}^{1} = \frac{4}{5} - \frac{7}{3} - 5 = -\frac{98}{15}.$

ii. $\int_0^\pi \sin 3x\,dx = \left(-\frac{1}{3}\cos 3x\right)\Big|_{x=0}^{\pi} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}.$

iii. $\int_{-1}^1 2xe^{5x^2+2}\,dx = \frac{1}{5}e^{5x^2+2}\Big|_{x=-1}^{1} = 0.$

II. Quiz zero problems (due at your lab on 08/30/2023)

Name: __________________________________ __________________________________

PS ID #: __________________________________ ______________________________

1. Find limits if possible

   $$\lim_{x\to 0^-} \frac{|x|}{x} =$$

   $$\lim_{x\to 0^+} \frac{|x|}{x} =$$

   $$\lim_{x\to 0} \frac{|x|}{x} =$$

2. Which ones among the functions, $\cos x$, $\sin x$, e^x, $\sqrt{x}$, $\frac{1}{x}$, are not defined on the entire real line?

3. Find an equation for the line passing through points $(2,3)$ and $(0,4)$. (hint: The line passing through points (x_0, y_0) and (x_1, y_1) is $y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$.)

4. Find the center and radius of the circle $x^2 - 3x + y^2 + y = 4$.

5. Given $f(x) = \sin(2x) + \cos(x^2 - 3x + 4)$, find $f'(x)$ and $f''(x)$.

6. Given $f(x) = e^{2x^3 - x}(4 - 5x + 6x^4)$, find $f'(x)$.

7. Given $f(x) = \dfrac{4 - 5x + \cos 4x}{x - 1}$, find $f'(x)$.

8. Evaluate $\int (4 - 5x + 6x^4)\, dx$.

9. Evaluate $\int xe^{x^2+3}\,dx$.

10. Evaluate $\int_0^2 (2t+2)\sin(t^2+2t+1)\,dt$.

11. Evaluate $\int_{-2}^{2} \sqrt{t^2+2t+1}\,dt$.