11 DIFFERENTIATION

This chapter gives several rules that will greatly simplify the task of finding the derivative of a function, thus enabling us to study how fast one quantity is changing with respect to another in many real-world situations. For example, we will be able to find how fast the population of an endangered species of whales grows after certain conservation measures have been implemented, how fast an economy's consumer price index (CPI) is changing at any time, and how fast the learning time changes with respect to the length of a list. We also see how these rules of differentiation facilitate the study of marginal analysis, the study of the rate of change of economic quantities. Finally, we introduce the notion of the differential of a function. Using differentials is a relatively easy way of approximating the change in one quantity due to a small change in a related quantity.

How is a pond's oxygen content affected by organic waste? In Example 7, page 712, you will see how to find the rate at which oxygen is being restored to the pond after organic waste has been dumped into it.

11.1 Basic Rules of Differentiation

Four Basic Rules

The method used in Chapter 10 for computing the derivative of a function is based on a faithful interpretation of the definition of the derivative as the limit of a quotient. Thus, to find the rule for the derivative f' of a function f, we first computed the difference quotient

$$\frac{f(x+h)-f(x)}{h}$$

and then evaluated its limit as h approached zero. As you have probably observed, this method is tedious even for relatively simple functions.

The main purpose of this chapter is to derive certain rules that will simplify the process of finding the derivative of a function. Throughout this book we will use the notation

$$\frac{d}{dx}[f(x)]$$

[read "d, d x of f of x"] to mean "the derivative of f with respect to x at x." In stating the rules of differentiation, we assume that the functions f and g are differentiable.

Rule 1: Derivative of a Constant

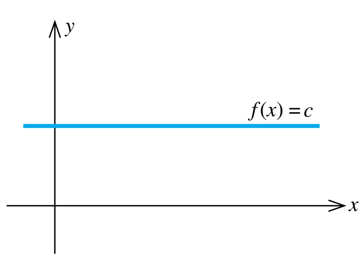
$$\frac{d}{dx}(c) = 0 \qquad (c\text{, a constant})$$

The derivative of a constant function is equal to zero.

We can see this from a geometric viewpoint by recalling that the graph of a constant function is a straight line parallel to the x-axis (Figure 11.1). Since the tangent line to a straight line at any point on the line coincides with the straight line itself, its slope [as given by the derivative of $f(x) = c$] must be zero. We can also use the definition of the derivative to prove this result by computing

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{c-c}{h} \\ &= \lim_{h\to 0} 0 = 0 \end{aligned}$$

FIGURE 11.1
The slope of the tangent line to the graph of $f(x) = c$, where c is a constant, is zero.



EXAMPLE 1

a. If $f(x) = 28$, then

$$f'(x) = \frac{d}{dx}(28) = 0$$

b. If $f(x) = -2$, then

$$f'(x) = \frac{d}{dx}(-2) = 0$$

Rule 2: The Power Rule

If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$.

Let's verify the power rule for the special case $n = 2$. If $f(x) = x^2$, then

$$\begin{aligned} f'(x) = \frac{d}{dx}(x^2) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x + h) = 2x \end{aligned}$$

as we set out to show.

The proof of the power rule for the general case is not easy to prove and will be omitted. However, you will be asked to prove the rule for the special case $n = 3$ in Exercise 72, page 706.

EXAMPLE 2

a. If $f(x) = x$, then

$$f'(x) = \frac{d}{dx}(x) = 1 \cdot x^{1-1} = x^0 = 1$$

b. If $f(x) = x^8$, then

$$f'(x) = \frac{d}{dx}(x^8) = 8x^7$$

c. If $f(x) = x^{5/2}$, then

$$f'(x) = \frac{d}{dx}(x^{5/2}) = \frac{5}{2}x^{3/2}$$

To differentiate a function whose rule involves a radical, we first rewrite the rule using fractional powers. The resulting expression can then be differentiated using the power rule.

EXAMPLE 3 Find the derivative of the following functions:

a. $f(x) = \sqrt{x}$ **b.** $g(x) = \dfrac{1}{\sqrt[3]{x}}$

SOLUTION ✔ **a.** Rewriting $\sqrt{x}$ in the form $x^{1/2}$, we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^{1/2}) \\ &= \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

b. Rewriting $\dfrac{1}{\sqrt[3]{x}}$ in the form $x^{-1/3}$, we obtain

$$\begin{aligned} g'(x) &= \frac{d}{dx}(x^{-1/3}) \\ &= -\frac{1}{3}x^{-4/3} = -\frac{1}{3x^{4/3}} \end{aligned}$$

■■■■

Rule 3: Derivative of a Constant Multiple of a Function

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)] \qquad (c, \text{ a constant})$$

The derivative of a constant times a differentiable function is equal to the constant times the derivative of the function.

This result follows from the following computations.

If $g(x) = cf(x)$, then

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h} \\ &= c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= cf'(x) \end{aligned}$$

EXAMPLE 4

a. If $f(x) = 5x^3$, then

$$f'(x) = \frac{d}{dx}(5x^3) = 5\frac{d}{dx}(x^3)$$
$$= 5(3x^2) = 15x^2$$

b. If $f(x) = \dfrac{3}{\sqrt{x}}$, then

$$f'(x) = \frac{d}{dx}(3x^{-1/2})$$
$$= 3\left(-\frac{1}{2}x^{-3/2}\right) = -\frac{3}{2x^{3/2}}$$

■■■■

Rule 4: The Sum Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

The derivative of the sum (difference) of two differentiable functions is equal to the sum (difference) of their derivatives.

This result may be extended to the sum and difference of any finite number of differentiable functions. Let's verify the rule for a sum of two functions.

If $s(x) = f(x) + g(x)$, then

$$s'(x) = \lim_{h\to 0}\frac{s(x+h) - s(x)}{h}$$
$$= \lim_{h\to 0}\frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$
$$= \lim_{h\to 0}\frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$

EXAMPLE 5

Find the derivatives of the following functions:

a. $f(x) = 4x^5 + 3x^4 - 8x^2 + x + 3$ **b.** $g(t) = \dfrac{t^2}{5} + \dfrac{5}{t^3}$

SOLUTION ✔

a. $f'(x) = \dfrac{d}{dx}(4x^5 + 3x^4 - 8x^2 + x + 3)$

$$= \frac{d}{dx}(4x^5) + \frac{d}{dx}(3x^4) - \frac{d}{dx}(8x^2) + \frac{d}{dx}(x) + \frac{d}{dx}(3)$$

$$= 20x^4 + 12x^3 - 16x + 1$$

b. Here, the independent variable is t instead of x, so we differentiate with respect to t. Thus,

$$g'(t) = \frac{d}{dt}\left(\frac{1}{5}t^2 + 5t^{-3}\right) \qquad \left(\text{Rewriting } \frac{1}{t^3} \text{ as } t^{-3}\right)$$

$$= \frac{2}{5}t - 15t^{-4}$$

$$= \frac{2t^5 - 75}{5t^4} \qquad \left(\text{Rewriting } t^{-4} \text{ as } \frac{1}{t^4} \text{ and simplifying}\right)$$

■■■■

EXAMPLE 6 Find the slope and an equation of the tangent line to the graph of $f(x) = 2x + 1/\sqrt{x}$ at the point (1, 3).

SOLUTION ✔ The slope of the tangent line at any point on the graph of f is given by

$$f'(x) = \frac{d}{dx}\left(2x + \frac{1}{\sqrt{x}}\right)$$

$$= \frac{d}{dx}(2x + x^{-1/2}) \qquad \left(\text{Rewriting } \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}} = x^{-1/2}\right)$$

$$= 2 - \frac{1}{2}x^{-3/2} \qquad \text{(Using the sum rule)}$$

$$= 2 - \frac{1}{2x^{3/2}}$$

In particular, the slope of the tangent line to the graph of f at (1, 3) (where $x = 1$) is

$$f'(1) = 2 - \frac{1}{2(1^{3/2})} = 2 - \frac{1}{2} = \frac{3}{2}$$

Using the point-slope form of the equation of a line with slope 3/2 and the point (1, 3), we see that an equation of the tangent line is

$$y - 3 = \frac{3}{2}(x - 1) \qquad [(y - y_1) = m(x - x_1)]$$

or, upon simplification,

$$y = \frac{3}{2}x + \frac{3}{2}$$

■■■■

APPLICATIONS

EXAMPLE 7 A group of marine biologists at the Neptune Institute of Oceanography recommended that a series of conservation measures be carried out over the next decade to save a certain species of whale from extinction. After implementing the conservation measures, the population of this species is expected to be

$$N(t) = 3t^3 + 2t^2 - 10t + 600 \qquad (0 \le t \le 10)$$

where $N(t)$ denotes the population at the end of year t. Find the rate of growth of the whale population when $t = 2$ and $t = 6$. How large will the whale population be 8 years after implementing the conservation measures?

SOLUTION ✔ The rate of growth of the whale population at any time t is given by

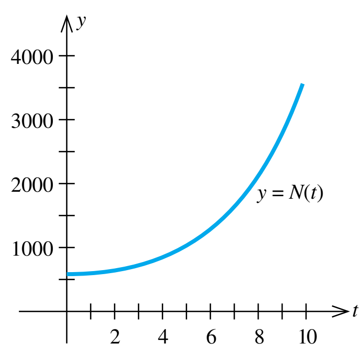
$$N'(t) = 9t^2 + 4t - 10$$

In particular, when $t = 2$ and $t = 6$, we have

$$N'(2) = 9(2)^2 + 4(2) - 10 = 34$$

$$N'(6) = 9(6)^2 + 4(6) - 10 = 338$$

FIGURE 11.2
The whale population after year t is given by $N(t)$.



so the whale population's rate of growth will be 34 whales per year after 2 years and 338 per year after 6 years.

The whale population at the end of the eighth year will be

$$N(8) = 3(8)^3 + 2(8)^2 - 10(8) + 600 = 2184 \text{ whales}$$

The graph of the function N appears in Figure 11.2. Note the rapid growth of the population in the later years, as the conservation measures begin to pay off, compared with the growth in the early years. ■■■■

EXAMPLE 8 The altitude of a rocket (in feet) t seconds into flight is given by

$$s = f(t) = -t^3 + 96t^2 + 195t + 5 \qquad (t \ge 0)$$

a. Find an expression v for the rocket's velocity at any time t.
b. Compute the rocket's velocity when $t = 0, 30, 50, 65$, and 70. Interpret your results.
c. Using the results from the solution to part (b) and the observation that at the highest point in its trajectory the rocket's velocity is zero, find the maximum altitude attained by the rocket.

SOLUTION ✔

a. The rocket's velocity at any time t is given by

$$v = f'(t) = -3t^2 + 192t + 195$$

b. The rocket's velocity when $t = 0$, 30, 50, 65, and 70 is given by

$$f'(0) = -3(0)^2 + 192(0) + 195 = 195$$
$$f'(30) = -3(30)^2 + 192(30) + 195 = 3255$$
$$f'(50) = -3(50)^2 + 192(50) + 195 = 2295$$
$$f'(65) = -3(65)^2 + 192(65) + 195 = 0$$
$$f'(70) = -3(70)^2 + 192(70) + 195 = -1065$$

or 195, 3255, 2295, 0, and -1065 feet per second (ft/sec).

Thus, the rocket has an initial velocity of 195 ft/sec at $t = 0$ and accelerates to a velocity of 3255 ft/sec at $t = 30$. Fifty seconds into the flight, the rocket's velocity is 2295 ft/sec, which is less than the velocity at $t = 30$. This means that the rocket begins to decelerate after an initial period of acceleration. (Later on we will learn how to determine the rocket's maximum velocity.)

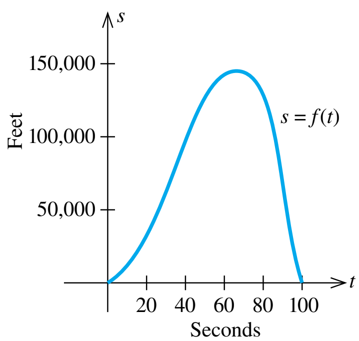
The deceleration continues: The velocity is 0 ft/sec at $t = 65$ and -1065 ft/sec when $t = 70$. This number tells us that 70 seconds into flight the rocket is heading back to Earth with a speed of 1065 ft/sec.

c. The results of part (b) show that the rocket's velocity is zero when $t = 65$. At this instant, the rocket's maximum altitude is

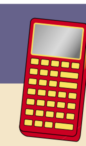
$$\begin{aligned} s = f(65) &= -(65)^3 + 96(65)^2 + 195(65) + 5 \\ &= 143{,}655 \text{ feet} \end{aligned}$$

A sketch of the graph of f appears in Figure 11.3. ■■■■

FIGURE 11.3
The rocket's altitude t seconds into flight is given by $f(t)$.



Exploring with Technology



Refer to Example 8.

1. Use a graphing utility to plot the graph of the velocity function
$$v = f'(t) = -3t^2 + 192t + 195$$
using the viewing rectangle $[0, 120] \times [-5000, 5000]$. Then, using ZOOM and TRACE or the root-finding capability of your graphing utility, verify that $f'(65) = 0$.
2. Plot the graph of the position function of the rocket
$$s = f(t) = -t^3 + 96t^2 + 195t + 5$$
using the viewing rectangle $[0, 120] \times [0, 150{,}000]$. Then, using ZOOM and TRACE repeatedly, verify that the maximum altitude of the rocket is 143,655 feet.
3. Use ZOOM and TRACE or the root-finding capability of your graphing utility to find when the rocket returns to Earth.

SELF-CHECK EXERCISES 11.1

1. Find the derivative of each of the following functions using the rules of differentiation.
   a. $f(x) = 1.5x^2 + 2x^{1.5}$
   b. $g(x) = 2\sqrt{x} + \dfrac{3}{\sqrt{x}}$
2. Let $f(x) = 2x^3 - 3x^2 + 2x - 1$.
   a. Compute $f'(x)$.
   b. What is the slope of the tangent line to the graph of f when $x = 2$?
   c. What is the rate of change of the function f at $x = 2$?
3. A certain country's gross domestic product (GDP) (in millions of dollars) is described by the function

$$G(t) = -2t^3 + 45t^2 + 20t + 6000 \qquad (0 \le t \le 11)$$

   where $t = 0$ corresponds to the beginning of 1990.
   a. At what rate was the GDP changing at the beginning of 1995? At the beginning of 1997? At the beginning of 2000?
   b. What was the average rate of growth of the GDP over the period 1995–2000?

Solutions to Self-Check Exercises 3.1 can be found on page 707.

11.1 Exercises

In Exercises 1–34, find the derivative of the function *f* by using the rules of differentiation.

1. $f(x) = -3$
2. $f(x) = 365$
3. $f(x) = x^5$
4. $f(x) = x^7$
5. $f(x) = x^{2.1}$
6. $f(x) = x^{0.8}$
7. $f(x) = 3x^2$
8. $f(x) = -2x^3$
9. $f(r) = \pi r^2$
10. $f(r) = \frac{4}{3}\pi r^3$
11. $f(x) = 9x^{1/3}$
12. $f(x) = \frac{5}{4}x^{4/5}$
13. $f(x) = 3\sqrt{x}$
14. $f(u) = \dfrac{2}{\sqrt{u}}$
15. $f(x) = 7x^{-12}$
16. $f(x) = 0.3x^{-1.2}$
17. $f(x) = 5x^2 - 3x + 7$
18. $f(x) = x^3 - 3x^2 + 1$
19. $f(x) = -x^3 + 2x^2 - 6$
20. $f(x) = x^4 - 2x^2 + 5$
21. $f(x) = 0.03x^2 - 0.4x + 10$
22. $f(x) = 0.002x^3 - 0.05x^2 + 0.1x - 20$
23. $f(x) = \dfrac{x^3 - 4x^2 + 3}{x}$
24. $f(x) = \dfrac{x^3 + 2x^2 + x - 1}{x}$
25. $f(x) = 4x^4 - 3x^{5/2} + 2$
26. $f(x) = 5x^{4/3} - \dfrac{2}{3}x^{3/2} + x^2 - 3x + 1$
27. $f(x) = 3x^{-1} + 4x^{-2}$
28. $f(x) = -\dfrac{1}{3}(x^{-3} - x^6)$
29. $f(t) = \dfrac{4}{t^4} - \dfrac{3}{t^3} + \dfrac{2}{t}$
30. $f(x) = \dfrac{5}{x^3} - \dfrac{2}{x^2} - \dfrac{1}{x} + 200$
31. $f(x) = 2x - 5\sqrt{x}$
32. $f(t) = 2t^2 + \sqrt{t^3}$
33. $f(x) = \dfrac{2}{x^2} - \dfrac{3}{x^{1/3}}$
34. $f(x) = \dfrac{3}{x^3} + \dfrac{4}{\sqrt{x}} + 1$
35. Let $f(x) = 2x^3 - 4x$. Find:
    a. $f'(-2)$ b. $f'(0)$ c. $f'(2)$
36. Let $f(x) = 4x^{5/4} + 2x^{3/2} + x$. Find:
    a. $f'(0)$ b. $f'(16)$

(continued on p. 704)

Using Technology

Finding the Rate of Change of a Function

We can use the numerical derivative operation of a graphing utility to obtain the value of the derivative at a given value of x. Since the derivative of a function $f(x)$ measures the rate of change of the function with respect to x, the numerical derivative operation can be used to answer questions pertaining to the rate of change of one quantity y with respect to another quantity x, where $y = f(x)$, for a specific value of x.

EXAMPLE 1 Let $y = 3t^3 + 2\sqrt{t}$.

a. Use the numerical derivative operation of a graphing utility to find how fast y is changing with respect to t when $t = 1$.
b. Verify the result of part (a), using the rules of differentiation of this section.

SOLUTION ✔

a. Write $f(t) = 3t^3 + 2\sqrt{t}$. Using the numerical derivative operation of a graphing utility, we find that the rate of change of y with respect to t when $t = 1$ is given by $f'(1) = 10$.
b. Here, $f(t) = 3t^3 + 2t^{1/2}$ and

$$f'(t) = 9t^2 + 2\left(\frac{1}{2}t^{-1/2}\right) = 9t^2 + \frac{1}{\sqrt{t}}$$

Using this result, we see that when $t = 1$, y is changing at the rate of

$$f'(1) = 9(1^2) + \frac{1}{\sqrt{1}} = 10$$

units per unit change in t, as obtained earlier. ■■■■

EXAMPLE 2 According to the U.S. Department of Energy and the Shell Development Company, a typical car's fuel economy depends on the speed it is driven and is approximated by the function

$$f(x) = 0.00000310315x^4 - 0.000455174x^3 + 0.00287869x^2 + 1.25986x \qquad (0 \le x \le 75)$$

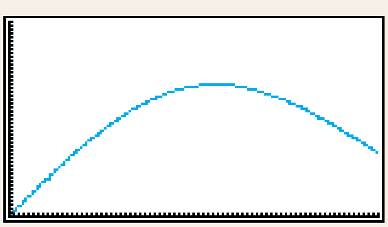
where x is measured in mph and $f(x)$ is measured in miles per gallon (mpg).

a. Use a graphing utility to graph the function f on the interval [0, 75].
b. Find the rate of change of f when $x = 20$ and $x = 50$.
c. Interpret your results.

Source: U.S. Department of Energy and the Shell Development Company

SOLUTION ✔

FIGURE T1



a. The result is shown in Figure T1.
b. Using the numerical derivative operation of a graphing utility, we see that $f'(20) = 0.9280996$. The rate of change of f when $x = 50$ is given by $f'(50) = -0.314501$.
c. The results of part (b) tell us that when a typical car is being driven at 20 mph, its fuel economy increases at the rate of approximately 0.9 mpg per 1 mph increase in its speed. At a speed of 50 mph, its fuel economy decreases at the rate of approximately 0.3 mpg per 1 mph increase in its speed. ■■■■

Exercises

In Exercises 1–6, use the numerical derivative operation of a graphing utility to find the rate of change of f(x) at the given value of x. Give your answer accurate to four decimal places.

1. $f(x) = 4x^5 - 3x^3 + 2x^2 + 1;\ x = 0.5$

2. $f(x) = -x^5 + 4x^2 + 3;\ x = 0.4$

3. $f(x) = x - 2\sqrt{x};\ x = 3$

4. $f(x) = \dfrac{\sqrt{x} - 1}{x};\ x = 2$

5. $f(x) = x^{1/2} - x^{1/3};\ x = 1.2$

6. $f(x) = 2x^{5/4} + x;\ x = 2$

7. **Carbon Monoxide in the Atmosphere** The projected average global atmosphere concentration of carbon monoxide is approximated by the function

$$f(t) = 0.881443t^4 - 1.45533t^3 + 0.695876t^2 + 2.87801t + 293 \qquad (0 \le t \le 4)$$

where t is measured in 40-yr intervals, with $t = 0$ corresponding to the beginning of 1860 and $f(t)$ is measured in parts per million by volume.
a. Use a graphing utility to plot the graph of f in the viewing rectangle $[0, 4] \times [280, 400]$.
b. Use a graphing utility to estimate how fast the projected average global atmospheric concentration of carbon monoxide was changing at the beginning of the year 1900 ($t = 1$) and at the beginning of 2000 ($t = 3.5$).
Source: "Beyond the Limits," Meadows et al.

8. **Growth of HMOs** Based on data compiled by the Group Health Association of America, the number of people receiving their care in an HMO (Health Maintenance Organization) from the beginning of 1984 through 1994 is approximated by the function

$$f(t) = 0.0514t^3 - 0.853t^2 + 6.8147t + 15.6524 \qquad (0 \le t \le 11)$$

where $f(t)$ gives the number of people in millions and t is measured in years, with $t = 0$ corresponding to the beginning of 1984.
a. Use a graphing utility to plot the graph of f in the viewing window $[0, 12] \times [0, 80]$.
b. How fast was the number of people receiving their care in an HMO growing at the beginning of 1992?
Source: Group Health Association of America

9. **Home Sales** According to the Greater Boston Real Estate Board—Multiple Listing Service, the average number of days a single-family home remains for sale from listing to accepted offer is approximated by the function

$$f(t) = 0.0171911t^4 - 0.662121t^3 + 6.18083t^2 - 8.97086t + 53.3357 \qquad (0 \le t \le 10)$$

where t is measured in years, with $t = 0$ corresponding to the beginning of 1984.
a. Use a graphing utility to plot the graph of f in the viewing rectangle $[0, 12] \times [0, 120]$.
b. How fast was the average number of days a single-family home remained for sale from listing to accepted offer changing at the beginning of 1984 ($t = 0$)? At the beginning of 1988 ($t = 4$)?
Source: Greater Boston Real Estate Board—Multiple Listing Service

10. **Spread of HIV** The estimated number of children newly infected with HIV through mother-to-child contact worldwide is given by

$$f(t) = -0.2083t^3 + 3.0357t^2 + 44.0476t + 200.2857 \qquad (0 \le t \le 12)$$

where $f(t)$ is measured in thousands and t is measured in years with $t = 0$ corresponding to the beginning of 1990.
a. Use a graphing utility to plot the graph of f in the viewing rectangle $[0, 12] \times [0, 800]$.
b. How fast was the estimated number of children newly infected with HIV through mother-to-child contact worldwide increasing at the beginning of the year 2000?
Source: United Nations

11. **Manufacturing Capacity** Data obtained from the Federal Reserve shows that the annual change in manufacturing capacity between 1988 and 1994 is given by

$$f(t) = 0.0388889t^3 - 0.283333t^2 + 0.477778t + 2.04286 \qquad (0 \le t \le 6)$$

where $f(t)$ is a percentage and t is measured in years, with $t = 0$ corresponding to the beginning of 1988.
a. Use a graphing utility to plot the graph of f in the viewing rectangle $[0, 8] \times [0, 4]$.
b. How fast was $f(t)$ changing at the beginning of 1990 ($t = 2$)? At the beginning of 1992 ($t = 4$)?
Source: Federal Reserve

In Exercises 37–40, find the given limit by evaluating the derivative of a suitable function at an appropriate point.
Hint: Look at the definition of the derivative.

37. $\lim_{h \to 0} \frac{(1+h)^3 - 1}{h}$

38. $\lim_{x \to 1} \frac{x^5 - 1}{x - 1}$
Hint: Let $h = x - 1$.

39. $\lim_{h \to 0} \frac{3(2+h)^2 - (2+h) - 10}{h}$

40. $\lim_{t \to 0} \frac{1 - (1+t)^2}{t(1+t)^2}$

In Exercises 41–44, find the slope and an equation of the tangent line to the graph of the function *f* at the specified point.

41. $f(x) = 2x^2 - 3x + 4$; $(2, 6)$

42. $f(x) = -\frac{5}{3}x^2 + 2x + 2$; $\left(-1, -\frac{5}{3}\right)$

43. $f(x) = x^4 - 3x^3 + 2x^2 - x + 1$; $(1, 0)$

44. $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$; $\left(4, \frac{5}{2}\right)$

45. Let $f(x) = x^3$.
a. Find the point on the graph of f where the tangent line is horizontal.
b. Sketch the graph of f and draw the horizontal tangent line.

46. Let $f(x) = x^3 - 4x^2$. Find the point(s) on the graph of f where the tangent line is horizontal.

47. Let $f(x) = x^3 + 1$.
a. Find the point(s) on the graph of f where the slope of the tangent line is equal to 12.
b. Find the equation(s) of the tangent line(s) of part (a).
c. Sketch the graph of f showing the tangent line(s).

48. Let $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 6$. Find the values of x for which:
a. $f'(x) = -12$ **b.** $f'(x) = 0$
c. $f'(x) = 12$

49. Let $f(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - x^2$. Find the point(s) on the graph of f where the slope of the tangent line is equal to:
a. $-2x$ **b.** 0 **c.** $10x$

50. A straight line perpendicular to and passing through the point of tangency of the tangent line is called the *normal* to the curve. Find an equation of the tangent line and the normal to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$.

51. **Growth of a Cancerous Tumor** The volume of a spherical cancer tumor is given by the function

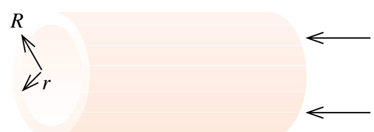
$$V(r) = \frac{4}{3}\pi r^3$$

where r is the radius of the tumor in centimeters. Find the rate of change in the volume of the tumor when:
a. $r = \frac{2}{3}$ cm **b.** $r = \frac{5}{4}$ cm

52. **Velocity of Blood in an Artery** The velocity (in centimeters per second) of blood r centimeters from the central axis of an artery is given by

$$v(r) = k(R^2 - r^2)$$

where k is a constant and R is the radius of the artery (see the accompanying figure). Suppose that $k = 1000$ and $R = 0.2$ cm. Find $v(0.1)$ and $v'(0.1)$ and interpret your results.



Blood vessel

53. **Effect of Stopping on Average Speed** According to data from a study by General Motors, the average speed of your trip A (in mph) is related to the number of stops per mile you make on the trip x by the equation

$$A = \frac{26.5}{x^{0.45}}$$

Compute dA/dx for $x = 0.25$ and $x = 2$ and interpret your results.
Source: General Motors

54. **Worker Efficiency** An efficiency study conducted for the Elektra Electronics Company showed that the number of "Space Commander" walkie-talkies assembled by the average worker t hr after starting work at 8 A.M. is given by

$$N(t) = -t^3 + 6t^2 + 15t$$

a. Find the rate at which the average worker will be assembling walkie-talkies t hr after starting work.
b. At what rate will the average worker be assembling walkie-talkies at 10 A.M.? At 11 A.M.?
c. How many walkie-talkies will the average worker assemble between 10 A.M. and 11 A.M.?

55. CONSUMER PRICE INDEX An economy's consumer price index (CPI) is described by the function

$$I(t) = -0.2t^3 + 3t^2 + 100 \qquad (0 \le t \le 10)$$

where $t = 0$ corresponds to 1990.
a. At what rate was the CPI changing in 1995? In 1997? In 2000?
b. What was the average rate of increase in the CPI over the period from 1995 to 2000?

56. EFFECT OF ADVERTISING ON SALES The relationship between the amount of money x that the Cannon Precision Instruments Corporation spends on advertising and the company's total sales $S(x)$ is given by the function

$$S(x) = -0.002x^3 + 0.6x^2 + x + 500 \qquad (0 \le x \le 200)$$

where x is measured in thousands of dollars. Find the rate of change of the sales with respect to the amount of money spent on advertising. Are Cannon's total sales increasing at a faster rate when the amount of money spent on advertising is (a) $100,000 or (b) $150,000?

57. POPULATION GROWTH A study prepared for a Sunbelt town's chamber of commerce projected that the town's population in the next 3 yr will grow according to the rule

$$P(t) = 50{,}000 + 30t^{3/2} + 20t$$

where $P(t)$ denotes the population t months from now. How fast will the population be increasing 9 mo and 16 mo from now?

58. CURBING POPULATION GROWTH Five years ago, the government of a Pacific island state launched an extensive propaganda campaign toward curbing the country's population growth. According to the Census Department, the population (measured in thousands of people) for the following 4 yr was

$$P(t) = -\frac{1}{3}t^3 + 64t + 3000$$

where t is measured in years and $t = 0$ at the start of the campaign. Find the rate of change of the population at the end of years 1, 2, 3, and 4. Was the plan working?

59. CONSERVATION OF SPECIES A certain species of turtle faces extinction because dealers collect truckloads of turtle eggs to be sold as aphrodisiacs. After severe conservation measures are implemented, it is hoped that the turtle population will grow according to the rule

$$N(t) = 2t^3 + 3t^2 - 4t + 1000 \qquad (0 \le t \le 10)$$

where $N(t)$ denotes the population at the end of year t. Find the rate of growth of the turtle population when $t = 2$ and $t = 8$. What will be the population 10 yr after the conservation measures are implemented?

60. FLIGHT OF A ROCKET The altitude (in feet) of a rocket t sec into flight is given by

$$s = f(t) = -2t^3 + 114t^2 + 480t + 1 \qquad (t \ge 0)$$

a. Find an expression v for the rocket's velocity at any time t.
b. Compute the rocket's velocity when $t = 0, 20, 40$, and 60. Interpret your results.
c. Using the results from the solution to part (b), find the maximum altitude attained by the rocket.
Hint: At its highest point, the velocity of the rocket is zero.

61. STOPPING DISTANCE OF A RACING CAR During a test by the editors of an auto magazine, the stopping distance s (in feet) of the MacPherson X-2 racing car conformed to the rule

$$s = f(t) = 120t - 15t^2 \qquad (t \ge 0)$$

where t was the time (in seconds) after the brakes were applied.
a. Find an expression for the car's velocity v at any time t.
b. What was the car's velocity when the brakes were first applied?
c. What was the car's stopping distance for that particular test?
Hint: The stopping time is found by setting $v = 0$.

62. DEMAND FUNCTIONS The demand function for the Luminar desk lamp is given by

$$p = f(x) = -0.1x^2 - 0.4x + 35$$

where x is the quantity demanded (measured in thousands) and p is the unit price in dollars.
a. Find $f'(x)$.
b. What is the rate of change of the unit price when the quantity demanded is 10,000 units ($x = 10$)? What is the unit price at that level of demand?

63. INCREASE IN TEMPORARY WORKERS According to the Labor Department, the number of temporary workers (in millions) is estimated to be

$$N(t) = 0.025t^2 + 0.255t + 1.505 \qquad (0 \le t \le 5)$$

where t is measured in years, with $t = 0$ corresponding to 1991.
a. How many temporary workers were there at the beginning of 1994?
b. How fast was the number of temporary workers growing at the beginning of 1994?
Source: Labor Department

64. **SALES OF DIGITAL SIGNAL PROCESSORS** The sales of digital signal processors (DSPs) in billions of dollars is projected to be

$$S(t) = 0.14t^2 + 0.68t + 3.1 \qquad (0 \le t \le 6)$$

where t is measured in years, with $t = 0$ corresponding to the beginning of 1997.
a. What were the sales of DSPs at the beginning of 1997? What will be the sales at the beginning of 2002?
b. How fast was the level of sales increasing at the beginning of 1997? How fast will the level of sales be increasing at the beginning of 2002?
Source: World Semiconductor Trade Statistics

65. **SUPPLY FUNCTIONS** The supply function for a certain make of transistor radio is given by

$$p = f(x) = 0.0001x^{5/4} + 10$$

where x is the quantity supplied and p is the unit price in dollars.
a. Find $f'(x)$.
b. What is the rate of change of the unit price if the quantity supplied is 10,000 transistor radios?

66. **PORTABLE PHONES** The percentage of the U.S. population with portable phones is projected to be

$$P(t) = 24.4t^{0.34} \qquad (1 \le t \le 10)$$

where t is measured in years, with $t = 1$ corresponding to the beginning of 1998.
a. What percentage of the U.S. population is expected to have portable phones by the beginning of 2006?
b. How fast is the percentage of the U.S. population with portable phones expected to be changing at the beginning of 2006?
Source: BancAmerica Robertson Stephens

67. **AVERAGE SPEED OF A VEHICLE ON A HIGHWAY** The average speed of a vehicle on a stretch of Route 134 between 6 A.M. and 10 A.M. on a typical weekday is approximated by the function

$$f(t) = 20t - 40\sqrt{t} + 50 \qquad (0 \le t \le 4)$$

where $f(t)$ is measured in miles per hour and t is measured in hours, $t = 0$ corresponding to 6 A.M.
a. Compute $f'(t)$.
b. Compute $f(0)$, $f(1)$, and $f(2)$ and interpret your results.
c. Compute $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ and interpret your results.

68. **HEALTH-CARE SPENDING** Despite efforts at cost containment, the cost of the Medicare program is increasing at a high rate. Two major reasons for this increase are an aging population and the constant development and extensive use by physicians of new technologies. Based on data from the Health Care Financing Administration and the U.S. Census Bureau, health-care spending through the year 2000 may be approximated by the function

$$S(t) = 0.02836t^3 - 0.05167t^2 + 9.60881t + 41.9 \qquad (0 \le t \le 35)$$

where $S(t)$ is the spending in billions of dollars and t is measured in years, with $t = 0$ corresponding to the beginning of 1965.
a. Find an expression for the rate of change of health-care spending at any time t.
b. How fast was health-care spending changing at the beginning of 1980? How fast was health-care spending changing at the beginning of 2000?
c. What was the amount of health-care spending at the beginning of 1980? What was the amount of health-care spending at the beginning of 2000?
Source: Health Care Financing Administration

69. **ON-LINE SHOPPING** Retail revenue per year from Internet shopping is approximated by the function

$$f(t) = 0.075t^3 + 0.025t^2 + 2.45t + 2.4 \qquad (0 \le t \le 4)$$

where $f(t)$ is measured in billions of dollars and t is measured in years with $t = 0$ corresponding to the beginning of 1997.
a. Find an expression giving the rate of change of the retail revenue per year from Internet shopping at any time t.
b. How fast was the retail revenue per year from Internet shopping changing at the beginning of the year 2000?
c. What was the retail revenue per year from Internet shopping at the beginning of the year 2000?
Source: Forrester Research, Inc.

In Exercises 70 and 71, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

70. If f and g are differentiable, then

$$\frac{d}{dx}[2f(x) - 5g(x)] = 2f'(x) - 5g'(x)$$

71. If $f(x) = \pi^x$, then $f'(x) = x\pi^{x-1}$

72. Prove the power rule (Rule 2) for the special case $n = 3$.

Hint: Compute $\lim_{h \to 0} \left[\frac{(x+h)^3 - x^3}{h}\right]$.

SOLUTIONS TO SELF-CHECK EXERCISES 11.1

1. **a.** $$f'(x) = \frac{d}{dx}(1.5x^2) + \frac{d}{dx}(2x^{1.5})$$
$$= (1.5)(2x) + (2)(1.5x^{0.5})$$
$$= 3x + 3\sqrt{x} = 3(x + \sqrt{x})$$

b. $$g'(x) = \frac{d}{dx}(2x^{1/2}) + \frac{d}{dx}(3x^{-1/2})$$
$$= (2)\left(\frac{1}{2}x^{-1/2}\right) + (3)\left(-\frac{1}{2}x^{-3/2}\right)$$
$$= x^{-1/2} - \frac{3}{2}x^{-3/2}$$
$$= \frac{1}{2}x^{-3/2}(2x - 3) = \frac{2x - 3}{2x^{3/2}}$$

2. **a.** $$f'(x) = \frac{d}{dx}(2x^3) - \frac{d}{dx}(3x^2) + \frac{d}{dx}(2x) - \frac{d}{dx}(1)$$
$$= (2)(3x^2) - (3)(2x) + 2$$
$$= 6x^2 - 6x + 2$$

b. The slope of the tangent line to the graph of f when $x = 2$ is given by
$$f'(2) = 6(2)^2 - 6(2) + 2 = 14$$

c. The rate of change of f at $x = 2$ is given by $f'(2)$. Using the results of part (b), we see that $f'(2)$ is 14 units/unit change in x.

3. **a.** The rate at which the GDP was changing at any time t $(0 < t < 11)$ is given by
$$G'(t) = -6t^2 + 90t + 20$$

In particular, the rates of change of the GDP at the beginning of the years 1995 $(t = 5)$, 1997 $(t = 7)$, and 2000 $(t = 10)$ are given by
$$G'(5) = 320, \quad G'(7) = 356, \quad \text{and} \quad G'(10) = 320$$

respectively—that is, by \$320 million/year, \$356 million/year, and \$320 million/year, respectively.

b. The average rate of growth of the GDP over the period from the beginning of 1995 $(t = 5)$ to the beginning of 2000 $(t = 10)$ is given by
$$\frac{G(10) - G(5)}{10 - 5} = \frac{[-2(10)^3 + 45(10)^2 + 20(10) + 6000]}{5} - \frac{[-2(5)^3 + 45(5)^2 + 20(5) + 6000]}{5}$$
$$= \frac{8700 - 6975}{5}$$

or \$345 million/year.

11.2 The Product and Quotient Rules

In this section we study two more rules of differentiation: the **product rule** and the **quotient rule.**

The Product Rule

The derivative of the product of two differentiable functions is given by the following rule:

Rule 5: The Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

The derivative of the product of two functions is the first function times the derivative of the second plus the second function times the derivative of the first.

The product rule may be extended to the case involving the product of any finite number of functions (see Exercise 63, page 717). We prove the product rule at the end of this section.

The derivative of the product of two functions is *not* given by the product of the derivatives of the functions; that is, in general

$$\frac{d}{dx}[f(x)g(x)] \neq f'(x)g'(x)$$

EXAMPLE 1 Find the derivative of the function

$$f(x) = (2x^2 - 1)(x^3 + 3)$$

SOLUTION ✔ By the product rule,

$$\begin{aligned} f'(x) &= (2x^2 - 1)\frac{d}{dx}(x^3 + 3) + (x^3 + 3)\frac{d}{dx}(2x^2 - 1) \\ &= (2x^2 - 1)(3x^2) + (x^3 + 3)(4x) \\ &= 6x^4 - 3x^2 + 4x^4 + 12x \\ &= 10x^4 - 3x^2 + 12x \\ &= x(10x^3 - 3x + 12) \end{aligned}$$

■■■■

EXAMPLE 2 Differentiate (that is, find the derivative of) the function

$$f(x) = x^3(\sqrt{x} + 1)$$

SOLUTION ✓ First, we express the function in exponential form, obtaining

$$f(x) = x^3(x^{1/2} + 1)$$

By the product rule,

$$\begin{aligned} f'(x) &= x^3 \frac{d}{dx}(x^{1/2} + 1) + (x^{1/2} + 1)\frac{d}{dx}x^3 \\ &= x^3\left(\frac{1}{2}x^{-1/2}\right) + (x^{1/2} + 1)(3x^2) \\ &= \frac{1}{2}x^{5/2} + 3x^{5/2} + 3x^2 \\ &= \frac{7}{2}x^{5/2} + 3x^2 \end{aligned}$$

REMARK We can also solve the problem by first expanding the product before differentiating f. Examples for which this is not possible will be considered in Section 11.3, where the true value of the product rule will be appreciated.

THE QUOTIENT RULE

The derivative of the quotient of two differentiable functions is given by the following rule:

Rule 6: The Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \qquad (g(x) \neq 0)$$

As an aid to remembering this expression, observe that it has the following form:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{(\text{Denominator})\begin{pmatrix}\text{Derivative of}\\ \text{numerator}\end{pmatrix} - (\text{Numerator})\begin{pmatrix}\text{Derivative of}\\ \text{denominator}\end{pmatrix}}{(\text{Square of denominator})}$$

For a proof of the quotient rule, see Exercise 64, page 717.

⚠ The derivative of a quotient is *not* equal to the quotient of the derivatives; that is,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}$$

For example, if $f(x) = x^3$ and $g(x) = x^2$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{d}{dx}(x) = 1$$

which is *not* equal to

$$\frac{f'(x)}{g'(x)} = \frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(x^2)} = \frac{3x^2}{2x} = \frac{3}{2}x$$

EXAMPLE 3 Find $f'(x)$ if $f(x) = \dfrac{x}{2x-4}$.

SOLUTION ✔ Using the quotient rule, we obtain

$$\begin{aligned} f'(x) &= \frac{(2x-4)\frac{d}{dx}(x) - x\frac{d}{dx}(2x-4)}{(2x-4)^2} \\ &= \frac{(2x-4)(1) - x(2)}{(2x-4)^2} \\ &= \frac{2x-4-2x}{(2x-4)^2} = -\frac{4}{(2x-4)^2} \end{aligned}$$

■■■■

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{x^2+1}{x^2-1}$.

SOLUTION ✔ By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2} \\ &= \frac{(x^2-1)(2x) - (x^2+1)(2x)}{(x^2-1)^2} \\ &= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2} \\ &= -\frac{4x}{(x^2-1)^2} \end{aligned}$$

■■■■

EXAMPLE 5 Find $h'(x)$ if

$$h(x) = \frac{\sqrt{x}}{x^2+1}$$

SOLUTION ✔ Rewrite $h(x)$ in the form $h(x) = \dfrac{x^{1/2}}{x^2+1}$. By the quotient rule, we find

$$\begin{aligned} h'(x) &= \frac{(x^2+1)\dfrac{d}{dx}(x^{1/2}) - x^{1/2}\dfrac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{(x^2+1)\left(\dfrac{1}{2}x^{-1/2}\right) - x^{1/2}(2x)}{(x^2+1)^2} \\ &= \frac{\dfrac{1}{2}x^{-1/2}(x^2+1-4x^2)}{(x^2+1)^2} \qquad \text{(Factoring out } \tfrac{1}{2}x^{-1/2} \text{ from the numerator)} \\ &= \frac{1-3x^2}{2\sqrt{x}(x^2+1)^2} \end{aligned}$$

APPLICATIONS

EXAMPLE 6 The sales (in millions of dollars) of a laser disc recording of a hit movie t years from the date of release is given by

$$S(t) = \frac{5t}{t^2+1}$$

a. Find the rate at which the sales are changing at time t.

b. How fast are the sales changing at the time the laser discs are released ($t = 0$)? Two years from the date of release?

SOLUTION ✔ **a.** The rate at which the sales are changing at time t is given by $S'(t)$. Using the quotient rule, we obtain

$$\begin{aligned} S'(t) &= \frac{d}{dt}\left[\frac{5t}{t^2+1}\right] = 5\frac{d}{dt}\left[\frac{t}{t^2+1}\right] \\ &= 5\left[\frac{(t^2+1)(1) - t(2t)}{(t^2+1)^2}\right] \\ &= 5\left[\frac{t^2+1-2t^2}{(t^2+1)^2}\right] = \frac{5(1-t^2)}{(t^2+1)^2} \end{aligned}$$

b. The rate at which the sales are changing at the time the laser discs are released is given by

$$S'(0) = \frac{5(1-0)}{(0+1)^2} = 5$$

That is, they are increasing at the rate of \$5 million per year.

Two years from the date of release, the sales are changing at the rate of

$$S'(2) = \frac{5(1-4)}{(4+1)^2} = -\frac{3}{5} = -0.6$$

That is, they are decreasing at the rate of $600,000 per year.

The graph of the function S is shown in Figure 11.4.

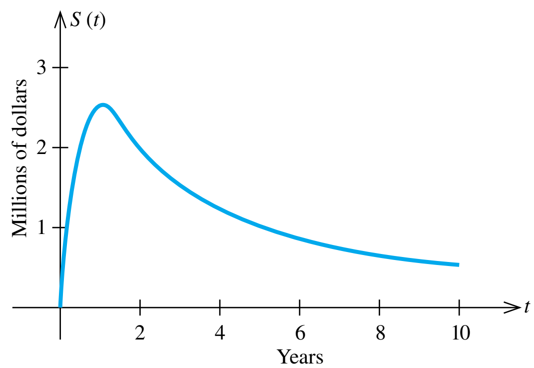


FIGURE 11.4
After a spectacular rise, the sales begin to taper off.

Exploring with Technology

Refer to Example 6.

1. Use a graphing utility to plot the graph of the function S using the viewing rectangle $[0, 10] \times [0, 3]$.
2. Use TRACE and ZOOM to determine the coordinates of the highest point on the graph of S in the interval $[0, 10]$. Interpret your results.

Group Discussion

Suppose the revenue of a company is given by $R(x) = xp(x)$, where x is the number of units of the product sold at a unit price of $p(x)$ dollars.
1. Compute $R'(x)$ and explain, in words, the relationship between $R'(x)$ and $p(x)$ and/or its derivative.
2. What can you say about $R'(x)$ if $p(x)$ is constant? Is this expected?

EXAMPLE 7 When organic waste is dumped into a pond, the oxidation process that takes place reduces the pond's oxygen content. However, given time, nature will restore the oxygen content to its natural level. Suppose the oxygen content t days after organic waste has been dumped into the pond is given by

$$f(t) = 100\left[\frac{t^2 + 10t + 100}{t^2 + 20t + 100}\right] \qquad (0 < t < \infty)$$

percent of its normal level.

a. Derive a general expression that gives the rate of change of the pond's oxygen level at any time t.
b. How fast is the pond's oxygen content changing 1 day, 10 days, and 20 days after the organic waste has been dumped?

SOLUTION ✓ **a.** The rate of change of the pond's oxygen level at any time t is given by the derivative of the function f. Thus, the required expression is

$$\begin{aligned} f'(t) &= 100\frac{d}{dt}\left[\frac{t^2+10t+100}{t^2+20t+100}\right] \\ &= 100\left[\frac{(t^2+20t+100)\frac{d}{dt}(t^2+10t+100)-(t^2+10t+100)\frac{d}{dt}(t^2+20t+100)}{(t^2+20t+100)^2}\right] \\ &= 100\left[\frac{(t^2+20t+100)(2t+10)-(t^2+10t+100)(2t+20)}{(t^2+20t+100)^2}\right] \\ &= 100\left[\frac{2t^3+10t^2+40t^2+200t+200t+1000-2t^3-20t^2-20t^2-200t-200t-2000}{(t^2+20t+100)^2}\right] \\ &= 100\left[\frac{10t^2-1000}{(t^2+20t+100)^2}\right] \end{aligned}$$

b. The rate at which the pond's oxygen content is changing 1 day after the organic waste has been dumped is given by

$$f'(1) = 100\left[\frac{10-1000}{(1+20+100)^2}\right] = -6.76$$

That is, it is dropping at the rate of 6.8% per day. After 10 days the rate is

$$f'(10) = 100\left[\frac{10(10)^2-1000}{(100+200+100)^2}\right] = 0$$

That is, it is neither increasing nor decreasing. After 20 days the rate is

$$f'(20) = 100\left[\frac{10(20)^2-1000}{(400+400+100)^2}\right] = 0.37$$

That is, the oxygen content is increasing at the rate of 0.37% per day, and the restoration process has indeed begun. ■■■■

Group Discussion

Consider a particle moving along a straight line. Newton's second law of motion states that the external force F acting on the particle is equal to the rate of change of its momentum. Thus,

$$F = \frac{d}{dt}(mv)$$

where m, the mass of the particle, and v, its velocity, are both functions of time t.

1. Use the product rule to show that

$$F = m\frac{dv}{dt} + v\frac{dm}{dt}$$

and explain the expression on the right-hand side in words.

2. Use the results of part 1 to show that if the mass of a particle is constant, then $F = ma$, where a is the acceleration of the particle.

Verification of the Product Rule

We will now verify the product rule. If $p(x) = f(x)g(x)$, then

$$\begin{aligned} p'(x) &= \lim_{h\to 0}\frac{p(x+h)-p(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \end{aligned}$$

By adding $-f(x+h)g(x) + f(x+h)g(x)$ (which is zero!) to the numerator and factoring, we have

$$\begin{aligned} p'(x) &= \lim_{h\to 0}\frac{f(x+h)[g(x+h)-g(x)] + g(x)[f(x+h)-f(x)]}{h} \\ &= \lim_{h\to 0}\left\{f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] + g(x)\left[\frac{f(x+h)-f(x)}{h}\right]\right\} \\ &= \lim_{h\to 0} f(x+h)\left[\frac{g(x+h)-g(x)}{h}\right] \\ &\quad + \lim_{h\to 0} g(x)\left[\frac{f(x+h)-f(x)}{h}\right] \qquad \text{(By Property 3 of limits)} \\ &= \lim_{h\to 0} f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &\quad + \lim_{h\to 0} g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad \text{(By Property 4 of limits)} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

Observe that in the second from the last link in the chain of equalities, we have used the fact that $\lim_{h\to 0} f(x+h) = f(x)$ because f is continuous at x.

SELF-CHECK EXERCISES 11.2

1. Find the derivative of $f(x) = \dfrac{2x+1}{x^2-1}$.

2. What is the slope of the tangent line to the graph of

$$f(x) = (x^2 + 1)(2x^3 - 3x^2 + 1)$$

at the point (2, 25)? How fast is the function f changing when $x = 2$?

3. The total sales of the Security Products Corporation in its first 2 yr of operation are given by

$$S = f(t) = \frac{0.3t^3}{1 + 0.4t^2} \qquad (0 \leq t \leq 2)$$

where S is measured in millions of dollars and $t = 0$ corresponds to the date Security Products began operations. How fast were the sales increasing at the beginning of the company's second year of operation?

Solutions to Self-Check Exercises 11.2 can be found on page 720.

11.2 Exercises

In Exercises 1–30, find the derivative of the given function.

1. $f(x) = 2x(x^2 + 1)$
2. $f(x) = 3x^2(x - 1)$
3. $f(t) = (t - 1)(2t + 1)$
4. $f(x) = (2x + 3)(3x - 4)$
5. $f(x) = (3x + 1)(x^2 - 2)$
6. $f(x) = (x + 1)(2x^2 - 3x + 1)$
7. $f(x) = (x^3 - 1)(x + 1)$
8. $f(x) = (x^3 - 12x)(3x^2 + 2x)$
9. $f(w) = (w^3 - w^2 + w - 1)(w^2 + 2)$
10. $f(x) = \frac{1}{5}x^5 + (x^2 + 1)(x^2 - x - 1) + 28$
11. $f(x) = (5x^2 + 1)(2\sqrt{x} - 1)$
12. $f(t) = (1 + \sqrt{t})(2t^2 - 3)$
13. $f(x) = (x^2 - 5x + 2)\left(x - \dfrac{2}{x}\right)$
14. $f(x) = (x^3 + 2x + 1)\left(2 + \dfrac{1}{x^2}\right)$
15. $f(x) = \dfrac{1}{x-2}$
16. $g(x) = \dfrac{3}{2x+4}$
17. $f(x) = \dfrac{x-1}{2x+1}$
18. $f(t) = \dfrac{1-2t}{1+3t}$
19. $f(x) = \dfrac{1}{x^2+1}$
20. $f(u) = \dfrac{u}{u^2+1}$
21. $f(s) = \dfrac{s^2-4}{s+1}$
22. $f(x) = \dfrac{x^3-2}{x^2+1}$
23. $f(x) = \dfrac{\sqrt{x}}{x^2+1}$
24. $f(x) = \dfrac{x^2+1}{\sqrt{x}}$
25. $f(x) = \dfrac{x^2+2}{x^2+x+1}$
26. $f(x) = \dfrac{x+1}{2x^2+2x+3}$
27. $f(x) = \dfrac{(x+1)(x^2+1)}{x-2}$
28. $f(x) = (3x^2 - 1)\left(x^2 - \dfrac{1}{x}\right)$
29. $f(x) = \dfrac{x}{x^2-4} - \dfrac{x-1}{x^2+4}$
30. $f(x) = \dfrac{x+\sqrt{3x}}{3x-1}$

In Exercises 31–34, suppose f and g are functions that are differentiable at $x = 1$ and that $f(1) = 2$, $f'(1) = -1$, $g(1) = -2$, $g'(1) = 3$. Find the value of $h'(1)$.

31. $h(x) = f(x)g(x)$
32. $h(x) = (x^2 + 1)g(x)$
33. $h(x) = \dfrac{xf(x)}{x+g(x)}$
34. $h(x) = \dfrac{f(x)g(x)}{f(x)-g(x)}$

In Exercises 35–38, find the derivative of each of the given functions and evaluate $f'(x)$ at the given value of x.

35. $f(x) = (2x - 1)(x^2 + 3)$; $x = 1$

36. $f(x) = \dfrac{2x + 1}{2x - 1}$; $x = 2$

37. $f(x) = \dfrac{x}{x^4 - 2x^2 - 1}$; $x = -1$

38. $f(x) = (\sqrt{x} + 2x)(x^{3/2} - x)$; $x = 4$

In Exercises 39–42, find the slope and an equation of the tangent line to the graph of the function f at the specified point.

39. $f(x) = (x^3 + 1)(x^2 - 2)$; $(2, 18)$

40. $f(x) = \dfrac{x^2}{x + 1}$; $\left(2, \dfrac{4}{3}\right)$

41. $f(x) = \dfrac{x + 1}{x^2 + 1}$; $(1, 1)$

42. $f(x) = \dfrac{1 + 2x^{1/2}}{1 + x^{3/2}}$; $\left(4, \dfrac{5}{9}\right)$

43. Find an equation of the tangent line to the graph of the function $f(x) = (x^3 + 1)(3x^2 - 4x + 2)$ at the point $(1, 2)$.

44. Find an equation of the tangent line to the graph of the function $f(x) = \dfrac{3x}{x^2 - 2}$ at the point $(2, 3)$.

45. Let $f(x) = (x^2 + 1)(2 - x)$. Find the point(s) on the graph of f where the tangent line is horizontal.

46. Let $f(x) = \dfrac{x}{x^2 + 1}$. Find the point(s) on the graph of f where the tangent line is horizontal.

47. Find the point(s) on the graph of the function $f(x) = (x^2 + 6)(x - 5)$ where the slope of the tangent line is equal to -2.

48. Find the point(s) on the graph of the function $f(x) = \dfrac{x + 1}{x - 1}$ where the slope of the tangent line is equal to $-1/2$.

49. A straight line perpendicular to and passing through the point of tangency of the tangent line is called the *normal* to the curve. Find the equation of the tangent line and the normal to the curve $y = \dfrac{1}{1 + x^2}$ at the point $(1, \frac{1}{2})$.

50. **Concentration of a Drug in the Bloodstream** The concentration of a certain drug in a patient's bloodstream t hr after injection is given by

$$C(t) = \frac{0.2t}{t^2 + 1}$$

a. Find the rate at which the concentration of the drug is changing with respect to time.
b. How fast is the concentration changing $\frac{1}{2}$ hr, 1 hr, and 2 hr after the injection?

51. **Cost of Removing Toxic Waste** A city's main well was recently found to be contaminated with trichloroethylene, a cancer-causing chemical, as a result of an abandoned chemical dump leaching chemicals into the water. A proposal submitted to the city's council members indicates that the cost, measured in millions of dollars, of removing x percent of the toxic pollutant is given by

$$C(x) = \frac{0.5x}{100 - x}$$

Find $C'(80)$, $C'(90)$, $C'(95)$, and $C'(99)$ and interpret your results.

52. **Drug Dosages** Thomas Young has suggested the following rule for calculating the dosage of medicine for children 1 to 12 yr old. If a denotes the adult dosage (in milligrams) and if t is the child's age (in years), then the child's dosage is given by

$$D(t) = \frac{at}{t + 12}$$

Suppose that the adult dosage of a substance is 500 mg. Find an expression that gives the rate of change of a child's dosage with respect to the child's age. What is the rate of change of a child's dosage with respect to his or her age for a 6-yr-old child? For a 10-yr-old child?

53. **Effect of Bactericide** The number of bacteria $N(t)$ in a certain culture t min after an experimental bactericide is introduced obeys the rule

$$N(t) = \frac{10{,}000}{1 + t^2} + 2000$$

Find the rate of change of the number of bacteria in the culture 1 minute after and 2 minutes after the bactericide is introduced. What is the population of the bacteria in the culture 1 min and 2 min after the bactericide is introduced?

54. **Demand Functions** The demand function for the Sicard wristwatch is given by

$$d(x) = \frac{50}{0.01x^2 + 1} \qquad (0 \le x \le 20)$$

where x (measured in units of a thousand) is the quantity demanded per week and $d(x)$ is the unit price in dollars.

a. Find $d'(x)$.
b. Find $d'(5)$, $d'(10)$, and $d'(15)$ and interpret your results.

55. LEARNING CURVES From experience, the Emory Secretarial School knows that the average student taking Advanced Typing will progress according to the rule

$$N(t) = \frac{60t + 180}{t + 6} \qquad (t \geq 0)$$

where $N(t)$ measures the number of words per minute the student can type after t wk in the course.
a. Find an expression for $N'(t)$.
b. Compute $N'(t)$ for $t = 1, 3, 4,$ and 7 and interpret your results.
c. Sketch the graph of the function N. Does it confirm the results obtained in part (b)?
d. What will be the average student's typing speed at the end of the 12-wk course?

56. BOX OFFICE RECEIPTS The total worldwide box office receipts for a long-running movie are approximated by the function

$$T(x) = \frac{120x^2}{x^2 + 4}$$

where $T(x)$ is measured in millions of dollars and x is the number of years since the movie's release. How fast are the total receipts changing 1 yr, 3 yr, and 5 yr after its release?

57. FORMALDEHYDE LEVELS A study on formaldehyde levels in 900 homes indicates that emissions of various chemicals can decrease over time. The formaldehyde level (parts per million) in an average home in the study is given by

$$f(t) = \frac{0.055t + 0.26}{t + 2} \qquad (0 \leq t \leq 12)$$

where t is the age of the house in years. How fast is the formaldehyde level of the average house dropping when it is new? At the beginning of its fourth year?
Source: Bonneville Power Administration

58. POPULATION GROWTH A major corporation is building a 4325-acre complex of homes, offices, stores, schools, and churches in the rural community of Glen Cove. As a result of this development, the planners have estimated that Glen Cove's population (in thousands) t yr from now will be given by

$$P(t) = \frac{25t^2 + 125t + 200}{t^2 + 5t + 40}$$

a. Find the rate at which Glen Cove's population is changing with respect to time.
b. What will be the population after 10 yr? At what rate will the population be increasing when $t = 10$?

In Exercises 59–62, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

59. If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x).$$

60. If f is differentiable, then

$$\frac{d}{dx}[xf(x)] = f(x) + xf'(x).$$

61. If f is differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{x^2}\right] = \frac{f'(x)}{2x}$$

62. If f, g, and h are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)g(x)}{h(x)}\right] = \frac{f'(x)g(x)h(x) + f(x)g'(x)h(x) - f(x)g(x)h'(x)}{[h(x)]^2}.$$

63. Extend the product rule for differentiation to the following case involving the product of three differentiable functions: Let $h(x) = u(x)v(x)w(x)$ and show that $h'(x) = u(x)v(x)w'(x) + u(x)v'(x)w(x) + u'(x)v(x)w(x)$.
Hint: Let $f(x) = u(x)v(x)$, $g(x) = w(x)$, and $h(x) = f(x)g(x)$, and apply the product rule to the function h.

64. Prove the quotient rule for differentiation (Rule 6).
Hint: Verify the following steps:

a. $$\frac{k(x+h) - k(x)}{h} = \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x+h)g(x)}$$

b. By adding $[-f(x)g(x) + f(x)g(x)]$ to the numerator and simplifying,

$$\frac{k(x+h) - k(x)}{h} = \frac{1}{g(x+h)g(x)} \times \left\{\left[\frac{f(x+h) - f(x)}{h}\right] \cdot g(x) - \left[\frac{g(x+h) - g(x)}{h}\right] \cdot f(x)\right\}$$

c. $$k'(x) = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Using Technology

THE PRODUCT AND QUOTIENT RULES

EXAMPLE 1 Let $f(x) = (2\sqrt{x} + 0.5x)\left(0.3x^3 + 2x - \frac{0.3}{x}\right)$. Find $f'(0.2)$.

SOLUTION ✔ Using the numerical derivative operation of a graphing utility, we find

$$f'(0.2) = 6.47974948127$$

EXAMPLE 2

Importance of Time in Treating Heart Attacks

According to the American Heart Association, the treatment benefit for heart attacks depends on the time to treatment and is described by the function

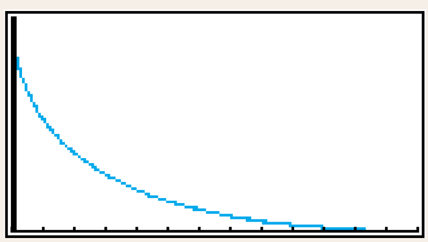
$$f(t) = \frac{-16.94t + 203.28}{t + 2.0328} \qquad (0 \le t \le 12)$$

where t is measured in hours and $f(t)$ is a percentage.

a. Use a graphing utility to graph the function f using the viewing rectangle $[0, 13] \times [0, 120]$.
b. Use a graphing utility to find the derivative of f when $t = 0$ and $t = 2$.
c. Interpret the results obtained in part (b).
Source: American Heart Association

SOLUTION ✔ **a.** The graph of f is shown in Figure T1.

FIGURE T1



b. Using the numerical derivative operation of a graphing utility, we find

$$f'(0) \approx -57.5266$$

$$f'(2) \approx -14.6165$$

c. The results of part (b) show that the treatment benefit drops off at the rate of 58% per hour at the time when the heart attack first occurs and falls off at the rate of 15% per hour when the time to treatment is 2 hours. Thus, it is extremely urgent that a patient suffering a heart attack receive medical attention as soon as possible. ■■■■

Exercises

In Exercises 1–6, use the numerical derivative operation of a graphing utility to find the rate of change of $f(x)$ at the given value of x. Give your answer accurate to four decimal places.

1. $f(x) = (2x^2 + 1)(x^3 + 3x + 4);\ x = -0.5$

2. $f(x) = (\sqrt{x} + 1)(2x^2 + x - 3);\ x = 1.5$

3. $f(x) = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1};\ x = 3$

4. $f(x) = \dfrac{\sqrt{x}(x^2 + 4)}{x^3 + 1};\ x = 4$

5. $f(x) = \dfrac{\sqrt{x}(1 + x^{-1})}{x + 1};\ x = 1$

6. $f(x) = \dfrac{x^2(2 + \sqrt{x})}{1 + \sqrt{x}};\ x = 1$

7. New Construction Jobs The president of a major housing construction company claims that the number of construction jobs created in the next t months is given by

$$f(t) = 1.42\left(\frac{7t^2 + 140t + 700}{3t^2 + 80t + 550}\right)$$

where $f(t)$ is measured in millions of jobs per year. At what rate will construction jobs be created 1 yr from now, assuming her projection is correct?

8. Population Growth A major corporation is building a 4325-acre complex of homes, offices, stores, schools, and churches in the rural community of Glen Cove. As a result of this development, the planners have estimated that Glen Cove's population (in thousands) t yr from now will be given by

$$P(t) = \frac{25t^2 + 125t + 200}{t^2 + 5t + 40}$$

a. What will be the population 10 yr from now?
b. At what rate will the population be increasing 10 yr from now?

SOLUTIONS TO SELF-CHECK EXERCISES 11.2

1. We use the quotient rule to obtain

$$\begin{aligned} f'(x) &= \frac{(x^2-1)\dfrac{d}{dx}(2x+1) - (2x+1)\dfrac{d}{dx}(x^2-1)}{(x^2-1)^2} \\ &= \frac{(x^2-1)(2) - (2x+1)(2x)}{(x^2-1)^2} \\ &= \frac{2x^2-2-4x^2-2x}{(x^2-1)^2} \\ &= \frac{-2x^2-2x-2}{(x^2-1)^2} \\ &= -\frac{2(x^2+x+1)}{(x^2-1)^2} \end{aligned}$$

2. The slope of the tangent line to the graph of f at any point is given by

$$\begin{aligned} f'(x) &= (x^2+1)\frac{d}{dx}(2x^3-3x^2+1) \\ &\quad + (2x^3-3x^2+1)\frac{d}{dx}(x^2+1) \\ &= (x^2+1)(6x^2-6x) + (2x^3-3x^2+1)(2x) \end{aligned}$$

In particular, the slope of the tangent line to the graph of f when $x = 2$ is

$$\begin{aligned} f'(2) &= (2^2+1)[6(2^2)-6(2)] \\ &\quad + [2(2^3)-3(2^2)+1][2(2)] \\ &= 60 + 20 = 80 \end{aligned}$$

Note that it is not necessary to simplify the expression for $f'(x)$ since we are required only to evaluate the expression at $x = 2$. We also conclude, from this result, that the function f is changing at the rate of 80 units/unit change in x when $x = 2$.

3. The rate at which the company's total sales are changing at any time t is given by

$$\begin{aligned} S'(t) &= \frac{(1+0.4t^2)\dfrac{d}{dt}(0.3t^3) - (0.3t^3)\dfrac{d}{dt}(1+0.4t^2)}{(1+0.4t^2)^2} \\ &= \frac{(1+0.4t^2)(0.9t^2) - (0.3t^3)(0.8t)}{(1+0.4t^2)^2} \end{aligned}$$

Therefore, at the beginning of the second year of operation, Security Products' sales were increasing at the rate of

$$S'(1) = \frac{(1+0.4)(0.9) - (0.3)(0.8)}{(1+0.4)^2} = 0.520408$$

or \$520,408/year.

11.3 The Chain Rule

This section introduces another rule of differentiation called the **chain rule.** When used in conjunction with the rules of differentiation developed in the last two sections, the chain rule enables us to greatly enlarge the class of functions we are able to differentiate.

The Chain Rule

Consider the function $h(x) = (x^2 + x + 1)^2$. If we were to compute $h'(x)$ using only the rules of differentiation from the previous sections, then our approach might be to expand $h(x)$. Thus,

$$\begin{aligned} h(x) &= (x^2 + x + 1)^2 = (x^2 + x + 1)(x^2 + x + 1) \\ &= x^4 + 2x^3 + 3x^2 + 2x + 1 \end{aligned}$$

from which we find

$$h'(x) = 4x^3 + 6x^2 + 6x + 2$$

But what about the function $H(x) = (x^2 + x + 1)^{100}$? The same technique may be used to find the derivative of the function H, but the amount of work involved in this case would be prodigious! Consider, also, the function $G(x) = \sqrt{x^2 + 1}$. For each of the two functions H and G, the rules of differentiation of the previous sections cannot be applied directly to compute the derivatives H' and G'.

Observe that both H and G are **composite functions;** that is, each is composed of, or built up from, simpler functions. For example, the function H is composed of the two simpler functions $f(x) = x^2 + x + 1$ and $g(x) = x^{100}$ as follows:

$$\begin{aligned} H(x) &= g[f(x)] = [f(x)]^{100} \\ &= (x^2 + x + 1)^{100} \end{aligned}$$

In a similar manner, we see that the function G is composed of the two simpler functions $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$. Thus,

$$\begin{aligned} G(x) &= g[f(x)] = \sqrt{f(x)} \\ &= \sqrt{x^2 + 1} \end{aligned}$$

As a first step toward finding the derivative h' of a composite function $h = g \circ f$ defined by $h(x) = g[f(x)]$, we write

$$u = f(x) \qquad \text{and} \qquad y = g[f(x)] = g(u)$$

The dependency of h on g and f is illustrated in Figure 11.5. Since u is a function of x, we may compute the derivative of u with respect to x, if f is a differentiable function, obtaining $du/dx = f'(x)$. Next, if g is a differentiable function of u, we may compute the derivative of g with respect to u, obtaining $dy/du = g'(u)$. Now, since the function h is composed of the function g and

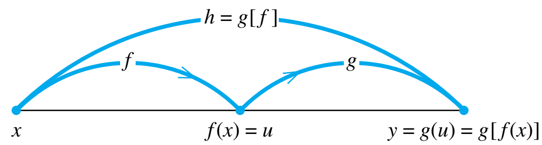


FIGURE 11.5
The composite function $h(x) = g[f(x)]$

the function f, we might suspect that the rule $h'(x)$ for the derivative h' of h will be given by an expression that involves the rules for the derivatives of f and g. But how do we combine these derivatives to yield h'?

This question can be answered by interpreting the derivative of each function as giving the rate of change of that function. For example, suppose $u = f(x)$ changes three times as fast as x—that is,

$$f'(x) = \frac{du}{dx} = 3$$

And suppose $y = g(u)$ changes twice as fast as u—that is,

$$g'(u) = \frac{dy}{du} = 2$$

Then, we would expect $y = h(x)$ to change six times as fast as x—that is,

$$h'(x) = g'(u)f'(x) = (2)(3) = 6$$

or equivalently,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (2)(3) = 6$$

This observation suggests the following result, which we state without proof.

Rule 7: The Chain Rule

If $h(x) = g[f(x)]$, then

$$h'(x) = \frac{d}{dx} g(f(x)) = g'(f(x))f'(x) \qquad \textbf{(1)}$$

Equivalently, if we write $y = h(x) = g(u)$, where $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \textbf{(2)}$$

REMARKS

1. If we label the composite function h in the following manner

Inside function
↓
$$h(x) = g[f(x)]$$
↑
Outside function

then $h'(x)$ is just the *derivative* of the "outside function" *evaluated at* the "inside function" times the *derivative* of the "inside function."

2. Equation (2) can be remembered by observing that if we "cancel" the du's, then

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx} = \frac{dy}{dx}$$

The Chain Rule for Powers of Functions

Many composite functions have the special form $h(x) = g(f(x))$, where g is defined by the rule $g(x) = x^n$ (n, a real number)—that is,

$$h(x) = [f(x)]^n$$

In other words, the function h is given by the power of a function f. The functions

$$h(x) = (x^2 + x + 1)^2, \qquad H = (x^2 + x + 1)^{100}, \qquad G = \sqrt{x^2 + 1}$$

discussed earlier are examples of this type of composite function. By using the following corollary of the chain rule, the general power rule, we can find the derivative of this type of function much more easily than by using the chain rule directly.

The General Power Rule

If the function f is differentiable and $h(x) = [f(x)]^n$ (n, a real number), then

$$h'(x) = \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x) \tag{3}$$

To see this, we observe that $h(x) = g(f(x))$, where $g(x) = x^n$, so that, by virtue of the chain rule, we have

$$\begin{aligned} h'(x) &= g'(f(x))f'(x) \\ &= n[f(x)]^{n-1}f'(x) \end{aligned}$$

since $g'(x) = nx^{n-1}$.

EXAMPLE 1 Find $H'(x)$, if

$$H(x) = (x^2 + x + 1)^{100}$$

SOLUTION ✔ Using the general power rule, we obtain

$$\begin{aligned} H'(x) &= 100(x^2 + x + 1)^{99}\frac{d}{dx}(x^2 + x + 1) \\ &= 100(x^2 + x + 1)^{99}(2x + 1) \end{aligned}$$

Observe the bonus that comes from using the chain rule: The answer is completely factored.

EXAMPLE 2 Differentiate the function $G(x) = \sqrt{x^2 + 1}$.

SOLUTION ✔ We rewrite the function $G(x)$ as

$$G(x) = (x^2 + 1)^{1/2}$$

and apply the general power rule, obtaining

$$\begin{aligned} G'(x) &= \frac{1}{2}(x^2 + 1)^{-1/2}\frac{d}{dx}(x^2 + 1) \\ &= \frac{1}{2}(x^2 + 1)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + 1}} \end{aligned}$$

■■■■

EXAMPLE 3 Differentiate the function $f(x) = x^2(2x + 3)^5$.

SOLUTION ✔ Applying the product rule followed by the general power rule, we obtain

$$\begin{aligned} f'(x) &= x^2\frac{d}{dx}(2x + 3)^5 + (2x + 3)^5\frac{d}{dx}(x^2) \\ &= (x^2)5(2x + 3)^4 \cdot \frac{d}{dx}(2x + 3) + (2x + 3)^5(2x) \\ &= 5x^2(2x + 3)^4(2) + 2x(2x + 3)^5 \\ &= 2x(2x + 3)^4(5x + 2x + 3) = 2x(7x + 3)(2x + 3)^4 \end{aligned}$$

■■■■

EXAMPLE 4 Find $f'(x)$ if $f(x) = \dfrac{1}{(4x^2 - 7)^2}$.

SOLUTION ✔ Rewriting $f(x)$ and then applying the general power rule, we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{1}{(4x^2 - 7)^2}\right] = \frac{d}{dx}(4x^2 - 7)^{-2} \\ &= -2(4x^2 - 7)^{-3}\frac{d}{dx}(4x^2 - 7) \\ &= -2(4x^2 - 7)^{-3}(8x) = -\frac{16x}{(4x^2 - 7)^3} \end{aligned}$$

■■■■

EXAMPLE 5 Find the slope of the tangent line to the graph of the function

$$f(x) = \left(\frac{2x + 1}{3x + 2}\right)^3$$

at the point $(0, \frac{1}{8})$.

SOLUTION ✓ The slope of the tangent line to the graph of f at any point is given by $f'(x)$. To compute $f'(x)$, we use the general power rule followed by the quotient rule, obtaining

$$\begin{aligned} f'(x) &= 3\left(\frac{2x+1}{3x+2}\right)^2 \frac{d}{dx}\left(\frac{2x+1}{3x+2}\right) \\ &= 3\left(\frac{2x+1}{3x+2}\right)^2 \left[\frac{(3x+2)(2)-(2x+1)(3)}{(3x+2)^2}\right] \\ &= 3\left(\frac{2x+1}{3x+2}\right)^2 \left[\frac{6x+4-6x-3}{(3x+2)^2}\right] \\ &= \frac{3(2x+1)^2}{(3x+2)^4} \end{aligned}$$

In particular, the slope of the tangent line to the graph of f at $(0, \frac{1}{8})$ is given by

$$f'(0) = \frac{3(0+1)^2}{(0+2)^4} = \frac{3}{16}$$

■■■■

Exploring with Technology

Refer to Example 5.

1. Use a graphing utility to plot the graph of the function f using the viewing rectangle $[-2, 1] \times [-1, 2]$. Then draw the tangent line to the graph of f at the point $(0, \frac{1}{8})$.
2. For a better picture, repeat part 1 using the viewing rectangle $[-1, 1] \times [-0.1, 0.3]$.
3. Use the numerical differentiation capability of the graphing utility to verify that the slope of the tangent line at $(0, \frac{1}{8})$ is $\frac{3}{16}$.

Applications

EXAMPLE 6 The membership of The Fitness Center, which opened a few years ago, is approximated by the function

$$N(t) = 100(64 + 4t)^{2/3} \qquad (0 \le t \le 52)$$

where $N(t)$ gives the number of members at the beginning of week t.

a. Find $N'(t)$.
b. How fast was the center's membership increasing initially $(t = 0)$?
c. How fast was the membership increasing at the beginning of the 40th week?
d. What was the membership when the center first opened? At the beginning of the 40th week?

SOLUTION ✓ **a.** Using the general power rule, we obtain

$$\begin{aligned} N'(t) &= \frac{d}{dt}[100(64+4t)^{2/3}] \\ &= 100\frac{d}{dt}(64+4t)^{2/3} \\ &= 100\left(\frac{2}{3}\right)(64+4t)^{-1/3}\frac{d}{dt}(64+4t) \\ &= \frac{200}{3}(64+4t)^{-1/3}(4) \\ &= \frac{800}{3(64+4t)^{1/3}} \end{aligned}$$

b. The rate at which the membership was increasing when the center first opened is given by

$$N'(0) = \frac{800}{3(64)^{1/3}} \approx 66.7$$

or approximately 67 people per week.

c. The rate at which the membership was increasing at the beginning of the 40th week is given by

$$N'(40) = \frac{800}{3(64+160)^{1/3}} \approx 43.9$$

or approximately 44 people per week.

d. The membership when the center first opened is given by

$$N(0) = 100(64)^{2/3} = 100(16)$$

or approximately 1600 people. The membership at the beginning of the 40th week is given by

$$N(40) = 100(64+160)^{2/3} \approx 3688.3$$

or approximately 3688 people.

Group Discussion

The profit P of a one-product software manufacturer depends on the number of units of its products sold. The manufacturer estimates that it will sell x units of its product per week. Suppose $P = g(x)$ and $x = f(t)$, where g and f are differentiable functions.

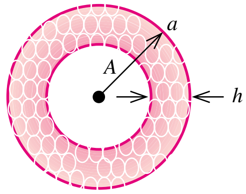
1. Write an expression giving the rate of change of the profit with respect to the number of units sold.

2. Write an expression giving the rate of change of the number of units sold per week.

3. Write an expression giving the rate of change of the profit per week.

EXAMPLE 7 Arteriosclerosis begins during childhood when plaque (soft masses of fatty material) forms in the arterial walls, blocking the flow of blood through the arteries and leading to heart attacks, strokes, and gangrene. Suppose the idealized cross section of the aorta is circular with radius a cm and by year t the thickness of the plaque (assume it is uniform) is $h = f(t)$ cm (Figure 11.6). Then the area of the opening is given by $A = \pi(a - h)^2$ square centimeters (cm^2).

FIGURE 11.6
Cross section of the aorta



Suppose the radius of an individual's artery is 1 cm ($a = 1$) and the thickness of the plaque in year t is given by

$$h = g(t) = 1 - 0.01(10{,}000 - t^2)^{1/2} \text{ cm}$$

Since the area of the arterial opening is given by

$$A = f(h) = \pi(1 - h)^2$$

the rate at which A is changing with respect to time is given by

$$\frac{dA}{dt} = \frac{dA}{dh} \cdot \frac{dh}{dt} = f'(h) \cdot g'(t) \quad \text{(By the chain rule)}$$

$$= 2\pi(1 - h)(-1)\left[-0.01\left(\frac{1}{2}\right)(10{,}000 - t^2)^{-1/2}(-2t)\right] \quad \text{(Using the chain rule twice)}$$

$$= -2\pi(1 - h)\left[\frac{0.01t}{(10{,}000 - t^2)^{1/2}}\right]$$

$$= -\frac{0.02\pi(1 - h)t}{\sqrt{10{,}000 - t^2}}$$

For example, when $t = 50$,

$$h = g(50) = 1 - 0.01(10{,}000 - 2500)^{1/2} \approx 0.134$$

so that

$$\frac{dA}{dt} = -\frac{0.02\pi(1 - 0.134)50}{\sqrt{10{,}000 - 2500}} \approx -0.03$$

That is, the area of the arterial opening is decreasing at the rate of 0.03 cm^2/year. ■■■■

Group Discussion

Suppose the population P of a certain bacteria culture is given by $P = f(T)$, where T is the temperature of the medium. Further, suppose the temperature T is a function of time t in seconds—that is, $T = g(t)$. Give an interpretation of each of the following quantities:

1. $\dfrac{dP}{dT}$ **2.** $\dfrac{dT}{dt}$ **3.** $\dfrac{dP}{dt}$ **4.** $(f \circ g)(t)$ **5.** $f'(g(t))g'(t)$

Using Technology

Finding the Derivative of a Composite Function

EXAMPLE 1 Find the rate of change of $f(x) = \sqrt{x}(1 + 0.02x^2)^{3/2}$ when $x = 2.1$.

SOLUTION Using the numerical derivative operation of a graphing utility, we find

$$f'(2.1) = 0.582146320119$$

or approximately 0.58 unit per unit change in x.

EXAMPLE 2 The management of Astro World ("The Amusement Park of the Future") estimates that the total number of visitors (in thousands) to the amusement park t hours after opening time at 9 A.M. is given by

$$N(t) = \frac{30t}{\sqrt{2 + t^2}}$$

What is the rate at which visitors are admitted to the amusement park at 10:30 A.M.?

SOLUTION Using the numerical derivative operation of a graphing utility, we find

$$N'(1.5) \approx 6.8481$$

or approximately 6848 visitors per hour.

Exercises

In Exercises 1–6, use the numerical derivative operation of a graphing utility to find the rate of change of $f(x)$ at the given value of x. Give your answer accurate to four decimal places.

1. $f(x) = \sqrt{x^2 - x^4}$; $x = 0.5$

2. $f(x) = x - \sqrt{1 - x^2}$; $x = 0.4$

3. $f(x) = x\sqrt{1 - x^2}$; $x = 0.2$

4. $f(x) = (x + \sqrt{x^2 + 4})^{3/2}$; $x = 1$

5. $f(x) = \dfrac{\sqrt{1 + x^2}}{x^3 + 2}$; $x = -1$

6. $f(x) = \dfrac{x^3}{1 + (1 + x^2)^{3/2}}$; $x = 3$

7. **Worldwide Production of Vehicles** According to *Automotive News,* the worldwide production of vehicles between 1960 and 1990 is given by the function

$$f(t) = 16.5\sqrt{1 + 2.2t} \qquad (0 \le t \le 3)$$

where $f(t)$ is measured in units of a million and t is measured in decades, with $t = 0$ corresponding to the beginning of 1960. What was the rate of change of the worldwide production of vehicles at the beginning of 1970? At the beginning of 1980?
Source: Automotive News

8. **Accumulation Years** People from their mid-40s to their mid-50s are in the prime investing years. Demographic studies of this type are of particular importance to financial institutions. The function

$$N(t) = 34.4(1 + 0.32125t)^{0.15} \qquad (0 \le t \le 12)$$

gives the projected number of people in this age group in the United States (in millions) in year t where $t = 0$ corresponds to the beginning of 1996.
a. How large is this segment of the population projected to be at the beginning of 2005?
b. How fast will this segment of the population be growing at the beginning of 2005?
Source: Census Bureau

Self-Check Exercises 11.3

1. Find the derivative of

$$f(x) = -\frac{1}{\sqrt{2x^2 - 1}}$$

2. Suppose the life expectancy at birth (in years) of a female in a certain country is described by the function

$$g(t) = 50.02(1 + 1.09t)^{0.1} \qquad (0 \le t \le 150)$$

where t is measured in years and $t = 0$ corresponds to the beginning of 1900.
a. What is the life expectancy at birth of a female born at the beginning of 1980? At the beginning of the year 2000?
b. How fast is the life expectancy at birth of a female born at any time t changing?

Solutions to Self-Check Exercises 11.3 can be found on page 733.

11.3 Exercises

In Exercises 1–46, find the derivative of the given function.

1. $f(x) = (2x - 1)^4$
2. $f(x) = (1 - x)^3$
3. $f(x) = (x^2 + 2)^5$
4. $f(t) = 2(t^3 - 1)^5$
5. $f(x) = (2x - x^2)^3$
6. $f(x) = 3(x^3 - x)^4$
7. $f(x) = (2x + 1)^{-2}$
8. $f(t) = \frac{1}{2}(2t^2 + t)^{-3}$
9. $f(x) = (x^2 - 4)^{3/2}$
10. $f(t) = (3t^2 - 2t + 1)^{3/2}$
11. $f(x) = \sqrt{3x - 2}$
12. $f(t) = \sqrt{3t^2 - t}$
13. $f(x) = \sqrt[3]{1 - x^2}$
14. $f(x) = \sqrt{2x^2 - 2x + 3}$
15. $f(x) = \dfrac{1}{(2x + 3)^3}$
16. $f(x) = \dfrac{2}{(x^2 - 1)^4}$
17. $f(t) = \dfrac{1}{\sqrt{2t - 3}}$
18. $f(x) = \dfrac{1}{\sqrt{2x^2 - 1}}$
19. $y = \dfrac{1}{(4x^4 + x)^{3/2}}$
20. $f(t) = \dfrac{4}{\sqrt[3]{2t^2 + t}}$
21. $f(x) = (3x^2 + 2x + 1)^{-2}$
22. $f(t) = (5t^3 + 2t^2 - t + 4)^{-3}$
23. $f(x) = (x^2 + 1)^3 - (x^3 + 1)^2$
24. $f(t) = (2t - 1)^4 + (2t + 1)^4$
25. $f(t) = (t^{-1} - t^{-2})^3$
26. $f(v) = (v^{-3} + 4v^{-2})^3$
27. $f(x) = \sqrt{x + 1} + \sqrt{x - 1}$
28. $f(u) = (2u + 1)^{3/2} + (u^2 - 1)^{-3/2}$
29. $f(x) = 2x^2(3 - 4x)^4$
30. $h(t) = t^2(3t + 4)^3$
31. $f(x) = (x - 1)^2(2x + 1)^4$
32. $g(u) = (1 + u^2)^5(1 - 2u^2)^8$
33. $f(x) = \left(\dfrac{x + 3}{x - 2}\right)^3$
34. $f(x) = \left(\dfrac{x + 1}{x - 1}\right)^5$
35. $s(t) = \left(\dfrac{t}{2t + 1}\right)^{3/2}$
36. $g(s) = \left(s^2 + \dfrac{1}{s}\right)^{3/2}$
37. $g(u) = \sqrt{\dfrac{u + 1}{3u + 2}}$
38. $g(x) = \sqrt{\dfrac{2x + 1}{2x - 1}}$
39. $f(x) = \dfrac{x^2}{(x^2 - 1)^4}$
40. $g(u) = \dfrac{2u^2}{(u^2 + u)^3}$
41. $h(x) = \dfrac{(3x^2 + 1)^3}{(x^2 - 1)^4}$
42. $g(t) = \dfrac{(2t - 1)^2}{(3t + 2)^4}$
43. $f(x) = \dfrac{\sqrt{2x + 1}}{x^2 - 1}$
44. $f(t) = \dfrac{4t^2}{\sqrt{2t^2 + 2t - 1}}$
45. $g(t) = \dfrac{\sqrt{t + 1}}{\sqrt{t^2 + 1}}$
46. $f(x) = \dfrac{\sqrt{x^2 + 1}}{\sqrt{x^2 - 1}}$

In Exercises 47–52, find $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{dx}$.

47. $y = u^{4/3}$ and $u = 3x^2 - 1$

48. $y = \sqrt{u}$ and $u = 7x - 2x^2$

49. $y = u^{-2/3}$ and $u = 2x^3 - x + 1$

50. $y = 2u^2 + 1$ and $u = x^2 + 1$

51. $y = \sqrt{u} + \frac{1}{\sqrt{u}}$ and $u = x^3 - x$

52. $y = \frac{1}{u}$ and $u = \sqrt{x} + 1$

53. Suppose $F(x) = g(f(x))$ and $f(2) = 3$, $f'(2) = -3$, $g(3) = 5$, and $g'(3) = 4$. Find $F'(2)$.

54. Suppose $h = f \circ g$. Find $h'(0)$ given that $f(0) = 6$, $f'(5) = -2$, $g(0) = 5$, and $g'(0) = 3$.

55. Suppose $F(x) = f(x^2 + 1)$. Find $F'(1)$ if $f'(2) = 3$.

56. Let $F(x) = f(f(x))$. Does it follow that $F'(x) = [f'(x)]^2$?
Hint: Let $f(x) = x^2$.

57. Suppose $h = g \circ f$. Does it follow that $h' = g' \circ f'$?
Hint: Let $f(x) = x$ and $g(x) = x^2$.

58. Suppose $h = f \circ g$. Show that $h' = (f' \circ g)g'$.

In Exercises 59–62, find an equation of the tangent line to the graph of the function at the given point.

59. $f(x) = (1 - x)(x^2 - 1)^2$; $(2, -9)$

60. $f(x) = \left(\frac{x+1}{x-1}\right)^2$; $(3, 4)$

61. $f(x) = x\sqrt{2x^2 + 7}$; $(3, 15)$

62. $f(x) = \frac{8}{\sqrt{x^2 + 6x}}$; $(2, 2)$

63. TELEVISION VIEWING The number of viewers of a television series introduced several years ago is approximated by the function

$$N(x) = (60 + 2x)^{2/3} \qquad (1 \le x \le 26)$$

where $N(x)$ (measured in millions) denotes the number of weekly viewers of the series in the xth week. Find the rate of increase of the weekly audience at the end of week 2 and at the end of week 12. How many viewers were there in week 2 and in week 24?

64. MALE LIFE EXPECTANCY Suppose the life expectancy of a male at birth in a certain country is described by the function

$$f(t) = 46.9(1 + 1.09t)^{0.1} \qquad (0 \le t \le 150)$$

where t is measured in years and $t = 0$ corresponds to the beginning of 1900. How long can a male born at the beginning of the year 2000 in that country expect to live? What is the rate of change of the life expectancy of a male born in that country at the beginning of the year 2000?

65. CONCENTRATION OF CARBON MONOXIDE IN THE AIR According to a joint study conducted by Oxnard's Environmental Management Department and a state government agency, the concentration of carbon monoxide in the air due to automobile exhaust t yr from now is given by

$$C(t) = 0.01(0.2t^2 + 4t + 64)^{2/3}$$

parts per million.
a. Find the rate at which the level of carbon monoxide is changing with respect to time.
b. Find the rate at which the level of carbon monoxide will be changing 5 yr from now.

66. CONTINUING EDUCATION ENROLLMENT The registrar of Kellogg University estimates that the total student enrollment in the Continuing Education division will be given by

$$N(t) = -\frac{20,000}{\sqrt{1 + 0.2t}} + 21,000$$

where $N(t)$ denotes the number of students enrolled in the division t yr from now. Find an expression for $N'(t)$. How fast is the student enrollment increasing currently? How fast will it be increasing 5 yr from now?

67. AIR POLLUTION According to the South Coast Air Quality Management District, the level of nitrogen dioxide, a brown gas that impairs breathing, present in the atmosphere on a certain May day in downtown Los Angeles is approximated by

$$A(t) = 0.03t^3(t - 7)^4 + 60.2 \qquad (0 \le t \le 7)$$

where $A(t)$ is measured in pollutant standard index and t is measured in hours, with $t = 0$ corresponding to 7 A.M.
a. Find $A'(t)$.
b. Find $A'(1)$, $A'(3)$, and $A'(4)$ and interpret your results.

68. EFFECT OF LUXURY TAX ON CONSUMPTION Government economists of a developing country determined that the purchase of imported perfume is related to a proposed "luxury tax" by the formula

$$N(x) = \sqrt{10{,}000 - 40x - 0.02x^2} \qquad (0 \le x \le 200)$$

where $N(x)$ measures the percentage of normal consumption of perfume when a "luxury tax" of x percent is imposed on it. Find the rate of change of $N(x)$ for taxes of 10%, 100%, and 150%.

69. Pulse Rate of an Athlete The pulse rate (the number of heartbeats per minute) of a long-distance runner t seconds after leaving the starting line is given by

$$P(t) = \frac{300\sqrt{\frac{1}{2}t^2 + 2t + 25}}{t + 25} \qquad (t \ge 0)$$

Compute $P'(t)$. How fast is the athlete's pulse rate increasing 10 sec, 60 sec, and 2 min into the run? What is her pulse rate 2 min into the run?

70. Thurstone Learning Model Psychologist L. L. Thurstone suggested the following relationship between learning time T and the length of a list n:

$$T = f(n) = An\sqrt{n - b}$$

where A and b are constants that depend on the person and the task.

a. Compute dT/dn and interpret your result.

b. For a certain person and a certain task, suppose $A = 4$ and $b = 4$. Compute $f'(13)$ and $f'(29)$ and interpret your results.

71. Oil Spills In calm waters, the oil spilling from the ruptured hull of a grounded tanker spreads in all directions. Assuming that the area polluted is a circle and that its radius is increasing at a rate of 2 ft/sec, determine how fast the area is increasing when the radius of the circle is 40 ft.

72. Arteriosclerosis Refer to Example 7, page 727. Suppose the radius of an individual's artery is 1 cm and the thickness of the plaque (in centimeters) t yr from now is given by

$$h = g(t) = \frac{0.5t^2}{t^2 + 10} \qquad (0 \le t \le 10)$$

How fast will the arterial opening be decreasing 5 yr from now?

73. Traffic Flow Opened in the late 1950s, the Central Artery in downtown Boston was designed to move 75,000 vehicles a day. The number of vehicles moved per day is approximated by the function

$$x = f(t) = 6.25t^2 + 19.75t + 74.75 \qquad (0 \le t \le 5)$$

where x is measured in thousands and t in decades, with $t = 0$ corresponding to the beginning of 1959. Suppose the average speed of traffic flow in mph is given by

$$S = g(x) = -0.00075x^2 + 67.5 \qquad (75 \le x \le 350)$$

where x has the same meaning as before. What was the rate of change of the average speed of traffic flow at the beginning of 1999? What was the average speed of traffic flow at that time?

74. Hotel Occupancy Rates The occupancy rate of the all-suite Wonderland Hotel, located near an amusement park, is given by the function

$$r(t) = \frac{10}{81}t^3 - \frac{10}{3}t^2 + \frac{200}{9}t + 60 \qquad (0 \le t \le 12)$$

where t is measured in months and $t = 0$ corresponds to the beginning of January. Management has estimated that the monthly revenue (in thousands of dollars per month) is approximated by the function

$$R(r) = -\frac{3}{5000}r^3 + \frac{9}{50}r^2 \qquad (0 \le r \le 100)$$

where r is the occupancy rate.

a. Find an expression that gives the rate of change of Wonderland's occupancy rate with respect to time.

b. Find an expression that gives the rate of change of Wonderland's monthly revenue with respect to the occupancy rate.

c. What is the rate of change of Wonderland's monthly revenue with respect to time at the beginning of January? At the beginning of June?

Hint: Use the chain rule to find $R'(r(0))r'(0)$ and $R'(r(6))r'(6)$.

75. Effect of Housing Starts on Jobs The president of a major housing construction firm claims that the number of construction jobs created is given by

$$N(x) = 1.42x$$

where x denotes the number of housing starts. Suppose the number of housing starts in the next t mo is expected to be

$$x(t) = \frac{7t^2 + 140t + 700}{3t^2 + 80t + 550}$$

million units/year. Find an expression that gives the rate at which the number of construction jobs will be created t mo from now. At what rate will construction jobs be created 1 yr from now?

76. Demand for PCs The quantity demanded per month, x, of a certain make of personal computer (PC) is related to the average unit price, p (in dollars), of PCs by the equation

$$x = f(p) = \frac{100}{9}\sqrt{810{,}000 - p^2}$$

It is estimated that t mo from now, the average price of a PC will be given by

$$p(t) = \frac{400}{1 + \frac{1}{8}\sqrt{t}} + 200 \qquad (0 \le t \le 60)$$

dollars. Find the rate at which the quantity demanded per month of the PCs will be changing 16 mo from now.

77. **Cruise Ship Bookings** The management of Cruise World, operators of Caribbean luxury cruises, expects that the percentage of young adults booking passage on their cruises in the years ahead will rise dramatically. They have constructed the following model, which gives the percentage of young adult passengers in year t:

$$p = f(t) = 50\left(\frac{t^2 + 2t + 4}{t^2 + 4t + 8}\right) \qquad (0 \le t \le 5)$$

Young adults normally pick shorter cruises and generally spend less on their passage. The following model gives an approximation of the average amount of money R (in dollars) spent per passenger on a cruise when the percentage of young adults is p:

$$R(p) = 1000\left(\frac{p + 4}{p + 2}\right)$$

Find the rate at which the price of the average passage will be changing 2 yr from now.

In Exercises 78–81, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

78. If f and g are differentiable and $h = f \circ g$, then $h'(x) = f'[g(x)]g'(x)$.

79. If f is differentiable and c is a constant, then

$$\frac{d}{dx}[f(cx)] = cf'(cx).$$

80. If f is differentiable, then

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$

81. If f is differentiable, then

$$\frac{d}{dx}\left[f\left(\frac{1}{x}\right)\right] = f'\left(\frac{1}{x}\right)$$

82. In Section 11.1 we proved that

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

for the special case when $n = 2$. Use the chain rule to show that

$$\frac{d}{dx}(x^{1/n}) = \frac{1}{n}x^{1/n-1}$$

for any nonzero integer n, assuming that $f(x) = x^{1/n}$ is differentiable.

Hint: Let $f(x) = x^{1/n}$ so that $[f(x)]^n = x$. Differentiate both sides with respect to x.

83. With the aid of Exercise 82, prove that

$$\frac{d}{dx}(x^r) = rx^{r-1}$$

for any rational number r.

Hint: Let $r = m/n$, where m and n are integers with $n \neq 0$, and write $x^r = (x^m)^{1/n}$.

Solutions to Self-Check Exercises 11.3

1. Rewriting, we have

$$f(x) = -(2x^2 - 1)^{-1/2}$$

Using the general power rule, we find

$$\begin{aligned} f'(x) &= -\frac{d}{dx}(2x^2 - 1)^{-1/2} \\ &= -\left(-\frac{1}{2}\right)(2x^2 - 1)^{-3/2}\frac{d}{dx}(2x^2 - 1) \\ &= \frac{1}{2}(2x^2 - 1)^{-3/2}(4x) \\ &= \frac{2x}{(2x^2 - 1)^{3/2}} \end{aligned}$$

2. a. The life expectancy at birth of a female born at the beginning of 1980 is given by

$$g(80) = 50.02[1 + 1.09(80)]^{0.1} \approx 78.29$$

or approximately 78 yr. Similarly, the life expectancy at birth of a female born at the beginning of the year 2000 is given by

$$g(100) = 50.02[1 + 1.09(100)]^{0.1} \approx 80.04$$

or approximately 80 yr.

b. The rate of change of the life expectancy at birth of a female born at any time t is given by $g'(t)$. Using the general power rule, we have

$$\begin{aligned} g'(t) &= 50.02\frac{d}{dt}(1 + 1.09t)^{0.1} \\ &= (50.02)(0.1)(1 + 1.09t)^{-0.9}\frac{d}{dt}(1 + 1.09t) \\ &= (50.02)(0.1)(1.09)(1 + 1.09t)^{-0.9} \\ &= 5.45218(1 + 1.09t)^{-0.9} \\ &= \frac{5.45218}{(1 + 1.09t)^{0.9}} \end{aligned}$$

11.4 Marginal Functions in Economics

Marginal analysis is the study of the rate of change of economic quantities. For example, an economist is not merely concerned with the value of an economy's gross domestic product (GDP) at a given time but is equally concerned with the rate at which it is growing or declining. In the same vein, a manufacturer is not only interested in the total cost corresponding to a certain level of production of a commodity but is also interested in the rate of change of the total cost with respect to the level of production, and so on. Let's begin with an example to explain the meaning of the adjective *marginal,* as used by economists.

Cost Functions

EXAMPLE 1 Suppose the total cost in dollars incurred per week by the Polaraire Company for manufacturing x refrigerators is given by the total cost function

$$C(x) = 8000 + 200x - 0.2x^2 \qquad (0 \leq x \leq 400)$$

a. What is the actual cost incurred for manufacturing the 251st refrigerator?
b. Find the rate of change of the total cost function with respect to x when $x = 250$.
c. Compare the results obtained in parts (a) and (b).

SOLUTION ✓

a. The actual cost incurred in producing the 251st refrigerator is the difference between the total cost incurred in producing the first 251 refrigerators and the total cost of producing the first 250 refrigerators:

$$\begin{aligned} C(251) - C(250) &= [8000 + 200(251) - 0.2(251)^2] \\ &\quad - [8000 + 200(250) - 0.2(250)^2] \\ &= 45{,}599.8 - 45{,}500 \\ &= 99.80 \end{aligned}$$

or \$99.80.

b. The rate of change of the total cost function C with respect to x is given by the derivative of C—that is, $C'(x) = 200 - 0.4x$. Thus, when the level of production is 250 refrigerators, the rate of change of the total cost with respect to x is given by

$$\begin{aligned} C'(250) &= 200 - 0.4(250) \\ &= 100 \end{aligned}$$

or \$100.

c. From the solution to part (a), we know that the actual cost for producing the 251st refrigerator is \$99.80. This answer is very closely approximated by the answer to part (b), \$100. To see why this is so, observe that the difference $C(251) - C(250)$ may be written in the form

$$\frac{C(251) - C(250)}{1} = \frac{C(250 + 1) - C(250)}{1} = \frac{C(250 + h) - C(250)}{h}$$

where $h = 1$. In other words, the difference $C(251) - C(250)$ is precisely the average rate of change of the total cost function C over the interval [250, 251], or, equivalently, the slope of the secant line through the points (250, 45,500) and (251, 45,599.8). However, the number $C'(250) = 100$ is the instantaneous rate of change of the total cost function C at $x = 250$, or, equivalently, the slope of the tangent line to the graph of C at $x = 250$.

Now when h is small, the average rate of change of the function C is a good approximation to the instantaneous rate of change of the function C, or, equivalently, the slope of the secant line through the points in question is a good approximation to the slope of the tangent line through the point in question. Thus, we may expect

$$\begin{aligned} C(251) - C(250) &= \frac{C(251) - C(250)}{1} = \frac{C(250 + h) - C(250)}{h} \\ &\approx \lim_{h \to 0} \frac{C(250 + h) - C(250)}{h} = C'(250) \end{aligned}$$

which is precisely the case in this example. ■■■■

The actual cost incurred in producing an additional unit of a certain commodity given that a plant is already at a certain level of operation is called the **marginal cost.** Knowing this cost is very important to management in their decision-making processes. As we saw in Example 1, the marginal cost is approximated by the rate of change of the total cost function evaluated at the appropriate point. For this reason, economists have defined the **marginal cost function** to be the derivative of the corresponding total cost function. In other words, if C is a total cost function, then the marginal cost function is defined to be its derivative C'. Thus, the adjective *marginal* is synonymous with *derivative of*.

EXAMPLE 2

A subsidiary of the Elektra Electronics Company manufactures a programmable pocket calculator. Management determined that the daily total cost of producing these calculators (in dollars) is given by

$$C(x) = 0.0001x^3 - 0.08x^2 + 40x + 5000$$

where x stands for the number of calculators produced.

a. Find the marginal cost function.
b. What is the marginal cost when $x = 200$, 300, 400, and 600?
c. Interpret your results.

SOLUTION ✔

a. The marginal cost function C' is given by the derivative of the total cost function C. Thus,

$$C'(x) = 0.0003x^2 - 0.16x + 40$$

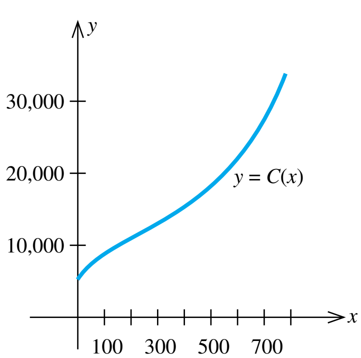
b. The marginal cost when $x = 200$, 300, 400, and 600 is given by

$$\begin{aligned} C'(200) &= 0.0003(200)^2 - 0.16(200) + 40 = 20 \\ C'(300) &= 0.0003(300)^2 - 0.16(300) + 40 = 19 \\ C'(400) &= 0.0003(400)^2 - 0.16(400) + 40 = 24 \\ C'(600) &= 0.0003(600)^2 - 0.16(600) + 40 = 52 \end{aligned}$$

or \$20, \$19, \$24, and \$52, respectively.

c. From the results of part (b), we see that Elektra's actual cost for producing the 201st calculator is approximately \$20. The actual cost incurred for producing one additional calculator when the level of production is already 300 calculators is approximately \$19, and so on. Observe that when the level of production is already 600 units, the actual cost of producing one additional unit is approximately \$52. The higher cost for producing this additional unit when the level of production is 600 units may be the result of several factors, among them excessive costs incurred because of overtime or higher maintenance, production breakdown caused by greater stress and strain on the equipment, and so on. The graph of the total cost function appears in Figure 11.7.

FIGURE 11.7
The cost of producing x calculators is given by $C(x)$.



AVERAGE COST FUNCTIONS

Let's now introduce another marginal concept closely related to the marginal cost. Let $C(x)$ denote the total cost incurred in producing x units of a certain commodity. Then the **average cost** of producing x units of the commodity is obtained by dividing the total production cost by the number of units produced. This leads to the following definition.

Average Cost Function

Suppose $C(x)$ is a total cost function. Then the **average cost function,** denoted by $\overline{C}(x)$ (read "C bar of x"), is

$$\frac{C(x)}{x} \quad \textbf{(4)}$$

The derivative $\overline{C}'(x)$ of the average cost function, called the **marginal average cost function,** measures the rate of change of the average cost function with respect to the number of units produced.

EXAMPLE 3 The total cost of producing x units of a certain commodity is given by

$$C(x) = 400 + 20x$$

dollars.

a. Find the average cost function $\overline{C}$.
b. Find the marginal average cost function $\overline{C}'$.
c. Interpret the results obtained in parts (a) and (b).

SOLUTION ✓ **a.** The average cost function is given by

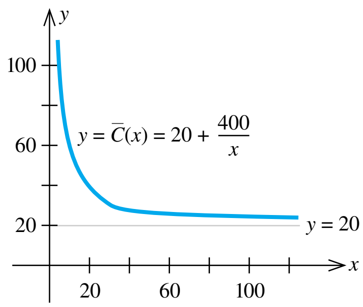
$$\overline{C}(x) = \frac{C(x)}{x} = \frac{400 + 20x}{x}$$
$$= 20 + \frac{400}{x}$$

b. The marginal average cost function is

$$\overline{C}'(x) = -\frac{400}{x^2}$$

c. Since the marginal average cost function is negative for all admissible values of x, the rate of change of the average cost function is negative for all $x > 0$; that is, $\overline{C}(x)$ decreases as x increases. However, the graph of $\overline{C}$ always lies

FIGURE 11.8
As the level of production increases, the average cost approaches $20.



above the horizontal line $y = 20$, but it approaches the line since

$$\lim_{x\to\infty} \overline{C}(x) = \lim_{x\to\infty}\left(20 + \frac{400}{x}\right) = 20$$

A sketch of the graph of the function $\overline{C}(x)$ appears in Figure 11.8. This result is fully expected if we consider the economic implications. Note that as the level of production increases, the fixed cost per unit of production, represented by the term $(400/x)$, drops steadily. The average cost approaches the constant unit of production, which is $20 in this case.

EXAMPLE 4 Once again consider the subsidiary of the Elektra Electronics Company. The daily total cost for producing its programmable calculators is given by

$$C(x) = 0.0001x^3 - 0.08x^2 + 40x + 5000$$

dollars, where x stands for the number of calculators produced (see Example 2).

a. Find the average cost function $\overline{C}$.
b. Find the marginal average cost function $\overline{C}'$. Compute $\overline{C}'(500)$.
c. Sketch the graph of the function $\overline{C}$ and interpret the results obtained in parts (a) and (b).

SOLUTION ✓ **a.** The average cost function is given by

$$\overline{C}(x) = \frac{C(x)}{x} = 0.0001x^2 - 0.08x + 40 + \frac{5000}{x}$$

b. The marginal average cost function is given by

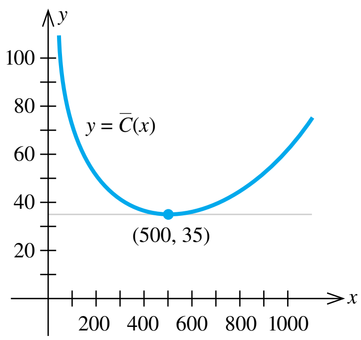
$$\overline{C}'(x) = 0.0002x - 0.08 - \frac{5000}{x^2}$$

Also,

$$\overline{C}'(500) = 0.0002(500) - 0.08 - \frac{5000}{(500)^2} = 0$$

c. To sketch the graph of the function $\overline{C}$, observe that if x is a small positive number, then $\overline{C}(x) > 0$. Furthermore, $\overline{C}(x)$ becomes arbitrarily large as x approaches zero from the right, since the term $(5000/x)$ becomes arbitrarily

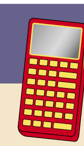
FIGURE 11.9
The average cost reaches a minimum of \$35 when 500 calculators are produced.



large as x approaches zero. Next, the result $\overline{C}'(500) = 0$ obtained in part (b) tells us that the tangent line to the graph of the function $\overline{C}$ is horizontal at the point (500, 35) on the graph. Finally, plotting the points on the graph corresponding to, say, $x = 100, 200, 300, \ldots, 900$, we obtain the sketch in Figure 11.9. As expected, the average cost drops as the level of production increases. But in this case, as opposed to the case in Example 3, the average cost reaches a minimum value of \$35, corresponding to a production level of 500, and *increases* thereafter.

This phenomenon is typical in situations where the marginal cost increases from some point on as production increases, as in Example 2. This situation is in contrast to that of Example 3, in which the marginal cost remains constant at any level of production. ■■■■

Exploring with Technology



Refer to Example 4.

1. Use a graphing utility to plot the graph of the average cost function

$$\overline{C}(x) = 0.0001x^2 - 0.08x + 40 + \frac{5000}{x}$$

using the viewing rectangle [0, 1000] × [0, 100]. Then, using **ZOOM** and **TRACE**, show that the lowest point on the graph of $\overline{C}$ is (500, 35).

2. Draw the tangent line to the graph of $\overline{C}$ at (500, 35). What is its slope? Is this expected?
3. Plot the graph of the marginal average cost function

$$\overline{C}'(x) = 0.0002x - 0.08 - \frac{5000}{x^2}$$

using the viewing rectangle [0, 2000] × [−1, 1]. Then use **ZOOM** and **TRACE** to show that the zero of the function $\overline{C}'$ occurs at $x = 500$. Verify this result using the root-finding capability of your graphing utility. Is this result compatible with that obtained in part (2)? Explain your answer.

REVENUE FUNCTIONS

Another marginal concept, the *marginal revenue function,* is associated with the revenue function R, given by

$$R(x) = px \tag{5}$$

where x is the number of units sold of a certain commodity and p is the unit selling price. In general, however, the unit selling price p of a commodity is related to the quantity x of the commodity demanded. This relationship, $p = f(x)$, is called a *demand equation* (see Section 10.3). Solving the demand equation for p in terms of x, we obtain the unit price function f, given by

$$p = f(x)$$

Thus, the revenue function R is given by

$$R(x) = px = xf(x)$$

where f is the unit price function. The derivative R' of the function R, called the **marginal revenue function,** measures the rate of change of the revenue function.

EXAMPLE 5 Suppose the relationship between the unit price p in dollars and the quantity demanded x of the Acrosonic model F loudspeaker system is given by the equation

$$p = -0.02x + 400 \qquad (0 \le x \le 20{,}000)$$

a. Find the revenue function R.
b. Find the marginal revenue function R'.
c. Compute $R'(2000)$ and interpret your result.

SOLUTION ✔ **a.** The revenue function R is given by

$$\begin{aligned} R(x) &= px \\ &= x(-0.02x + 400) \\ &= -0.02x^2 + 400x \qquad (0 \le x \le 20{,}000) \end{aligned}$$

b. The marginal revenue function R' is given by

$$R'(x) = -0.04x + 400$$

c. $$R'(2000) = -0.04(2000) + 400 = 320$$

Thus, the actual revenue to be realized from the sale of the 2001st loudspeaker system is approximately \$320. ■■■■

PROFIT FUNCTIONS

Our final example of a marginal function involves the profit function. The profit function P is given by

$$P(x) = R(x) - C(x) \qquad \textbf{(6)}$$

where R and C are the revenue and cost functions and x is the number of units of a commodity produced and sold. The **marginal profit function** $P'(x)$ measures the rate of change of the profit function P and provides us with a good approximation of the actual profit or loss realized from the sale of the $(x + 1)$st unit of the commodity (assuming the xth unit has been sold).

EXAMPLE 6 Refer to Example 5. Suppose the cost of producing x units of the Acrosonic model F loudspeaker is

$$C(x) = 100x + 200{,}000 \text{ dollars}$$

a. Find the profit function P.
b. Find the marginal profit function P'.
c. Compute $P'(2000)$ and interpret your result.
d. Sketch the graph of the profit function P.

SOLUTION ✔ **a.** From the solution to Example 5(a), we have

$$R(x) = -0.02x^2 + 400x$$

Thus, the required profit function P is given by

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-0.02x^2 + 400x) - (100x + 200{,}000) \\ &= -0.02x^2 + 300x - 200{,}000 \end{aligned}$$

b. The marginal profit function P' is given by

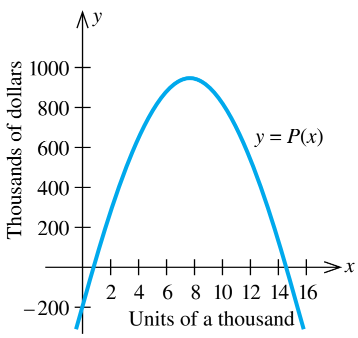
$$P'(x) = -0.04x + 300$$

c. $$P'(2000) = -0.04(2000) + 300 = 220$$

Thus, the actual profit realized from the sale of the 2001st loudspeaker system is approximately \$220.

d. The graph of the profit function P appears in Figure 11.10. ■■■■

FIGURE 11.10
The total profit made when x loudspeakers are produced is given by $P(x)$.

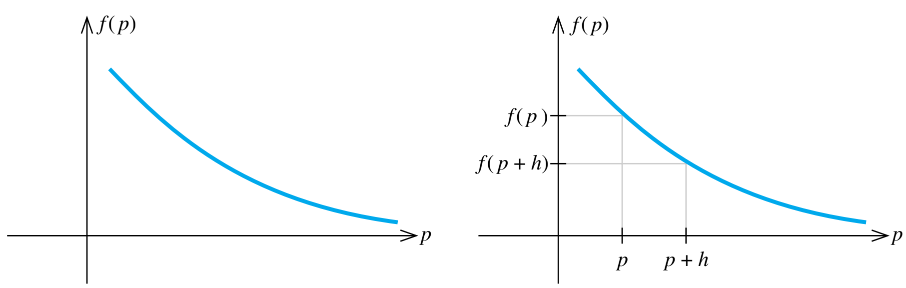


ELASTICITY OF DEMAND

Finally, let us use the marginal concepts introduced in this section to derive an important criterion used by economists to analyze the demand function: *elasticity of demand.*

In what follows, it will be convenient to write the demand function f in the form $x = f(p)$; that is, we will think of the quantity demanded of a certain commodity as a function of its unit price. Since the quantity demanded of a commodity usually decreases as its unit price increases, the function f is typically a decreasing function of p (Figure 11.11a).

FIGURE 11.11



(a) A demand function

(b) $f(p + h)$ is the quantity demanded when the unit price increases from p to $p + h$ dollars.

Suppose the unit price of a commodity is increased by h dollars from p dollars to $(p + h)$ dollars (Figure 11.11b). Then the quantity demanded drops from $f(p)$ units to $f(p + h)$ units, a change of $[f(p + h) - f(p)]$ units. The percentage change in the unit price is

$$\frac{h}{p}(100) \qquad \left(\frac{\text{Change in unit price}}{\text{Price } p}\right)(100)$$

and the corresponding percentage change in the quantity demanded is

$$100\left[\frac{f(p+h) - f(p)}{f(p)}\right] \qquad \left(\frac{\text{Change in quantity demanded}}{\text{Quantity demanded at price } p}\right)(100)$$

Now, one good way to measure the effect that a percentage change in price has on the percentage change in the quantity demanded is to look at the ratio of the latter to the former. We find

$$\frac{\text{Percentage change in the quantity demanded}}{\text{Percentage change in the unit price}} = \frac{100\left[\dfrac{f(p+h) - f(p)}{f(p)}\right]}{100\left(\dfrac{h}{p}\right)} = \frac{\dfrac{f(p+h) - f(p)}{h}}{\dfrac{f(p)}{p}}$$

If f is differentiable at p, then

$$\frac{f(p+h) - f(p)}{h} \approx f'(p)$$

when h is small. Therefore, if h is small, then the ratio is approximately equal to

$$\frac{f'(p)}{\dfrac{f(p)}{p}} = \frac{pf'(p)}{f(p)}$$

Economists call the negative of this quantity the *elasticity of demand.*

Elasticity of Demand

> If f is a differentiable demand function defined by $x = f(p)$, then the **elasticity of demand** at price p is given by
>
> $$E(p) = -\frac{pf'(p)}{f(p)} \qquad \textbf{(7)}$$

REMARK It will be shown later (Section 12.1) that if f is decreasing on an interval, then $f'(p) < 0$ for p in that interval. In light of this, we see that since both p and $f(p)$ are positive, the quantity $\dfrac{pf'(p)}{f(p)}$ is negative. Because economists would rather work with a positive value, the elasticity of demand $E(p)$ is defined to be the negative of this quantity.

EXAMPLE 7 Consider the demand equation

$$p = -0.02x + 400 \qquad (0 \le x \le 20{,}000)$$

which describes the relationship between the unit price in dollars and the quantity demanded x of the Acrosonic model F loudspeaker systems.

a. Find the elasticity of demand $E(p)$.
b. Compute $E(100)$ and interpret your result.
c. Compute $E(300)$ and interpret your result.

SOLUTION ✔ **a.** Solving the given demand equation for x in terms of p, we find

$$x = f(p) = -50p + 20{,}000$$

from which we see that

$$f'(p) = -50$$

Therefore,

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p(-50)}{-50p + 20{,}000} = \frac{p}{400 - p}$$

b. $E(100) = \dfrac{100}{400 - 100} = \dfrac{1}{3}$, which is the elasticity of demand when $p = 100$. To interpret this result, recall that $E(100)$ is the negative of the ratio of the percentage change in the quantity demanded to the percentage change in the unit price when $p = 100$. Therefore, our result tells us that when the unit price p is set at \$100 per speaker, an increase of 1% in the unit price will cause an increase of approximately 0.33% in the quantity demanded.

c. $E(300) = \dfrac{300}{400 - 300} = 3$, which is the elasticity of demand when $p = 300$. It tells us that when the unit price is set at \$300 per speaker, an increase of 1% in the unit price will cause a decrease of approximately 3% in the quantity demanded. ■■■■

Economists often use the following terminology to describe demand in terms of elasticity.

Elasticity of Demand

The demand is said to be **elastic** if $E(p) > 1$.
The demand is said to be **unitary** if $E(p) = 1$.
The demand is said to be **inelastic** if $E(p) < 1$.

As an illustration, our computations in Example 7 revealed that demand for Acrosonic loudspeakers is elastic when $p = 300$ but inelastic when $p = 100$. These computations confirm that when demand is elastic, a small percentage change in the unit price will result in a greater percentage change in the quantity demanded; and when demand is inelastic, a small percentage change in the unit price will cause a smaller percentage change in the quantity demanded. Finally, when demand is unitary, a small percentage change in the unit price will result in the same percentage change in the quantity demanded.

We can describe the way revenue responds to changes in the unit price using the notion of elasticity. If the quantity demanded of a certain commodity is related to its unit price by the equation $x = f(p)$, then the revenue realized through the sale of x units of the commodity at a price of p dollars each is

$$R(p) = px = pf(p)$$

The rate of change of the revenue with respect to the unit price p is given by

$$\begin{aligned} R'(p) &= f(p) + pf'(p) \\ &= f(p)\left[1 + \frac{pf'(p)}{f(p)}\right] \\ &= f(p)[1 - E(p)] \end{aligned}$$

Now, suppose demand is elastic when the unit price is set at a dollars. Then $E(a) > 1$, and so $1 - E(a) < 0$. Since $f(p)$ is positive for all values of p, we see that

$$R'(a) = f(a)[1 - E(a)] < 0$$

and so $R(p)$ is decreasing at $p = a$. This implies that a small increase in the unit price when $p = a$ results in a decrease in the revenue, whereas a small decrease in the unit price will result in an increase in the revenue. Similarly, you can show that if the demand is inelastic when the unit price is set at a dollars, then a small increase in the unit price will cause the revenue to increase, and a small decrease in the unit price will cause the revenue to decrease. Finally, if the demand is unitary when the unit price is set at a dollars, then $E(a) = 1$ and $R'(a) = 0$. This implies that a small increase or decrease in the unit price will not result in a change in the revenue. The following statements summarize this discussion.

1. If the demand is elastic at p ($E(p) > 1$), then an increase in the unit price will cause the revenue to decrease, whereas a decrease in the unit price will cause the revenue to increase.
2. If the demand is inelastic at p ($E(p) < 1$), then an increase in the unit price will cause the revenue to increase, and a decrease in the unit price will cause the revenue to decrease.
3. If the demand is unitary at p ($E(p) = 1$), then an increase in the unit price will cause the revenue to stay about the same.

These results are illustrated in Figure 11.12.

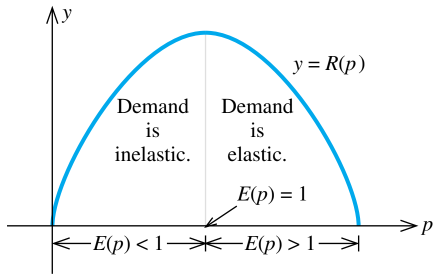


FIGURE 11.12
The revenue is increasing on an interval where the demand is inelastic, decreasing on an interval where the demand is elastic, and stationary at the point where the demand is unitary.

REMARK As an aid to remembering this, note the following:

1. If demand is elastic, then the change in revenue and the change in the unit price move in opposite directions.
2. If demand is inelastic, then they move in the same direction.

EXAMPLE 8 Refer to Example 7.

a. Is demand elastic, unitary, or inelastic when $p = 100$? When $p = 300$?
b. If the price is \$100, will raising the unit price slightly cause the revenue to increase or decrease?

SOLUTION ✓ **a.** From the results of Example 7, we see that $E(100) = \frac{1}{3} < 1$ and $E(300) = 3 > 1$. We conclude accordingly that demand is inelastic when $p = 100$ and elastic when $p = 300$.
b. Since demand is inelastic when $p = 100$, raising the unit price slightly will cause the revenue to increase.

SELF-CHECK EXERCISES 11.4

1. The weekly demand for Pulsar VCRs (videocassette recorders) is given by the demand equation

$$p = -0.02x + 300 \qquad (0 \le x \le 15{,}000)$$

where p denotes the wholesale unit price in dollars and x denotes the quantity demanded. The weekly total cost function associated with manufacturing these VCRs is

$$C(x) = 0.000003x^3 - 0.04x^2 + 200x + 70{,}000 \text{ dollars}$$

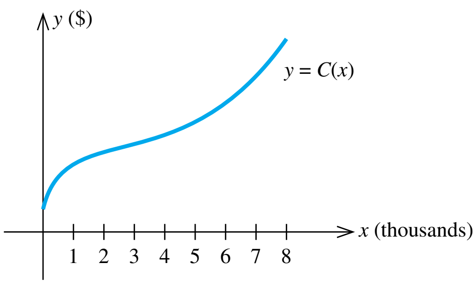
a. Find the revenue function R and the profit function P.
b. Find the marginal cost function C', the marginal revenue function R', and the marginal profit function P'.
c. Find the marginal average cost function $\overline{C}'$.
d. Compute $C'(3000)$, $R'(3000)$, and $P'(3000)$ and interpret your results.

2. Refer to the preceding exercise. Determine whether the demand is elastic, unitary, or inelastic when $p = 100$ and when $p = 200$.

Solutions to Self-Check Exercises 11.4 can be found on page 749.

11.4 Exercises

1. **Production Costs** The graph of a typical total cost function $C(x)$ associated with the manufacture of x units of a certain commodity is shown in the following figure.
a. Explain why the function C is always increasing.
b. As the level of production x increases, the cost per unit drops so that $C(x)$ increases but at a slower pace. However, a level of production is soon reached at which the cost per unit begins to increase dramatically (due to a shortage of raw material, overtime, breakdown of machinery due to excessive stress and strain) so that $C(x)$ continues to increase at a faster pace. Use the graph of C to find the approximate level of production x_0 where this occurs.



A calculator is recommended for Exercises 2–33.

2. **Marginal Cost** The total weekly cost (in dollars) incurred by the Lincoln Record Company in pressing x long-playing records is

$$C(x) = 2000 + 2x - 0.0001x^2 \qquad (0 \le x \le 6000)$$

a. What is the actual cost incurred in producing the 1001st and the 2001st record?
b. What is the marginal cost when $x = 1000$ and 2000?

3. **Marginal Cost** A division of Ditton Industries manufactures the Futura model microwave oven. The daily cost (in dollars) of producing these microwave ovens is

$$C(x) = 0.0002x^3 - 0.06x^2 + 120x + 5000$$

where x stands for the number of units produced.
a. What is the actual cost incurred in manufacturing the 101st oven? The 201st oven? The 301st oven?
b. What is the marginal cost when $x = 100$, 200, and 300?

4. **Marginal Average Cost** The Custom Office Company makes a line of executive desks. It is estimated that the total cost for making x units of their Senior Executive Model is

$$C(x) = 100x + 200{,}000$$

dollars per year.
a. Find the average cost function $\overline{C}$.
b. Find the marginal average cost function $\overline{C}'$.
c. What happens to $\overline{C}(x)$ when x is very large? Interpret your results.

5. **Marginal Average Cost** The management of the ThermoMaster Company, whose Mexican subsidiary manufactures an indoor–outdoor thermometer, has estimated that the total weekly cost (in dollars) for producing x thermometers is

$$C(x) = 5000 + 2x$$

a. Find the average cost function $\overline{C}$.
b. Find the marginal average cost function $\overline{C}'$.
c. Interpret your results.

6. Find the average cost function $\overline{C}$ and the marginal average cost function $\overline{C}'$ associated with the total cost function C of Exercise 2.

7. Find the average cost function $\overline{C}$ and the marginal average cost function $\overline{C}'$ associated with the total cost function C of Exercise 3.

8. **Marginal Revenue** The Williams Commuter Air Service realizes a monthly revenue of

$$R(x) = 8000x - 100x^2$$

dollars when the price charged per passenger is x dollars.
a. Find the marginal revenue R'.
b. Compute $R'(39)$, $R'(40)$, and $R'(41)$. What do your results imply?

9. **Marginal Revenue** The management of the Acrosonic Company plans to market the Electro-Stat, an electrostatic speaker system. The marketing department has determined that the demand for these speakers is

$$p = -0.04x + 800 \qquad (0 \le x \le 20{,}000)$$

where p denotes the speaker's unit price (in dollars) and x denotes the quantity demanded.
a. Find the revenue function R.
b. Find the marginal revenue function R'.
c. Compute $R'(5000)$ and interpret your result.

10. **MARGINAL PROFIT** Lynbrook West, an apartment complex, has 100 two-bedroom units. The monthly profit (in dollars) realized from renting x apartments is

$$P(x) = -10x^2 + 1760x - 50{,}000$$

a. What is the actual profit realized from renting the 51st unit, assuming that 50 units have already been rented?
b. Compute the marginal profit when $x = 50$ and compare your results with that obtained in part (a).

11. **MARGINAL PROFIT** Refer to Exercise 9. Acrosonic's production department estimates that the total cost (in dollars) incurred in manufacturing x Electro-Stat speaker systems in the first year of production will be

$$C(x) = 200x + 300{,}000$$

a. Find the profit function P.
b. Find the marginal profit function P'.
c. Compute $P'(5000)$ and $P'(8000)$.
d. Sketch the graph of the profit function and interpret your results.

12. **MARGINAL COST, REVENUE, AND PROFIT** The weekly demand for the Pulsar 25 color console television is

$$p = 600 - 0.05x \qquad (0 \le x \le 12{,}000)$$

where p denotes the wholesale unit price in dollars and x denotes the quantity demanded. The weekly total cost function associated with manufacturing the Pulsar 25 is given by

$$C(x) = 0.000002x^3 - 0.03x^2 + 400x + 80{,}000$$

where $C(x)$ denotes the total cost incurred in producing x sets.
a. Find the revenue function R and the profit function P.
b. Find the marginal cost function C', the marginal revenue function R', and the marginal profit function P'.
c. Compute $C'(2000)$, $R'(2000)$, and $P'(2000)$ and interpret your results.
d. Sketch the graphs of the functions C, R, and P and interpret parts (b) and (c) using the graphs obtained.

13. **MARGINAL COST, REVENUE, AND PROFIT** The Pulsar Corporation also manufactures a series of 19-inch color television sets. The quantity x of these sets demanded each week is related to the wholesale unit price p by the equation

$$p = -0.006x + 180$$

The weekly total cost incurred by Pulsar for producing x sets is

$$C(x) = 0.000002x^3 - 0.02x^2 + 120x + 60{,}000$$

dollars. Answer the questions in Exercise 12 for these data.

14. **MARGINAL AVERAGE COST** Find the average cost function $\overline{C}$ associated with the total cost function C of Exercise 12.
a. What is the marginal average cost function $\overline{C}'$?
b. Compute $\overline{C}'(5{,}000)$ and $\overline{C}'(10{,}000)$ and interpret your results.
c. Sketch the graph of $\overline{C}$.

15. **MARGINAL AVERAGE COST** Find the average cost function $\overline{C}$ associated with the total cost function C of Exercise 13.
a. What is the marginal average cost function $\overline{C}'$?
b. Compute $\overline{C}'(5{,}000)$ and $\overline{C}'(10{,}000)$ and interpret your results.

16. **MARGINAL REVENUE** The quantity of Sicard wristwatches demanded per month is related to the unit price by the equation

$$p = \frac{50}{0.01x^2 + 1} \qquad (0 \le x \le 20)$$

where p is measured in dollars and x in units of a thousand.
a. Find the revenue function R.
b. Find the marginal revenue function R'.
c. Compute $R'(2)$ and interpret your result.

17. **MARGINAL PROPENSITY TO CONSUME** The consumption function of the U.S. economy for 1929 to 1941 is

$$C(x) = 0.712x + 95.05$$

where $C(x)$ is the personal consumption expenditure and x is the personal income, both measured in billions of dollars. Find the rate of change of consumption with respect to income, dC/dx. This quantity is called the *marginal propensity to consume.*

18. **MARGINAL PROPENSITY TO CONSUME** Refer to Exercise 17. Suppose a certain economy's consumption function is

$$C(x) = 0.873x^{1.1} + 20.34$$

where $C(x)$ and x are measured in billions of dollars. Find the marginal propensity to consume when $x = 10$.

19. Marginal Propensity to Save Suppose $C(x)$ measures an economy's personal consumption expenditure and x the personal income, both in billions of dollars. Then,

$$S(x) = x - C(x) \qquad \text{(Income minus consumption)}$$

measures the economy's savings corresponding to an income of x billion dollars. Show that

$$\frac{dS}{dx} = 1 - \frac{dC}{dx}$$

The quantity dS/dx is called the *marginal propensity to save.*

20. Refer to Exercise 19. For the consumption function of Exercise 17, find the marginal propensity to save.

21. Refer to Exercise 19. For the consumption function of Exercise 18, find the marginal propensity to save when $x = 10$.

For each demand equation in Exercises 22–27, compute the elasticity of demand and determine whether the demand is elastic, unitary, or inelastic at the indicated price.

22. $x = -\frac{3}{2}p + 9;\ p = 2$

23. $x = -\frac{5}{4}p + 20;\ p = 10$

24. $x + \frac{1}{3}p - 20 = 0;\ p = 30$

25. $0.4x + p - 20 = 0;\ p = 10$

26. $p = 144 - x^2;\ p = 96$

27. $p = 169 - x^2;\ p = 29$

28. Elasticity of Demand The management of the Titan Tire Company has determined that the quantity demanded x of their Super Titan tires per week is related to the unit price p by the equation

$$x = \sqrt{144 - p}$$

where p is measured in dollars and x in units of a thousand.
a. Compute the elasticity of demand when $p = 63$, 96, and 108.
b. Interpret the results obtained in part (a).
c. Is the demand elastic, unitary, or inelastic when $p = 63$, 96, and 108?

29. Elasticity of Demand The demand equation for the Roland portable hair dryer is given by

$$x = \frac{1}{5}(225 - p^2) \qquad (0 \le p \le 15)$$

where x (measured in units of a hundred) is the quantity demanded per week and p is the unit price in dollars.
a. Is the demand elastic or inelastic when $p = 8$ and when $p = 10$?
b. When is the demand unitary?
Hint: Solve $E(p) = 1$ for p.
c. If the unit price is lowered slightly from \$10, will the revenue increase or decrease?
d. If the unit price is increased slightly from \$8, will the revenue increase or decrease?

30. Elasticity of Demand The quantity demanded per week x (in units of a hundred) of the Mikado miniature camera is related to the unit price p (in dollars) by the demand equation

$$x = \sqrt{400 - 5p} \qquad (0 \le p \le 80)$$

a. Is the demand elastic or inelastic when $p = 40$? When $p = 60$?
b. When is the demand unitary?
c. If the unit price is lowered slightly from \$60, will the revenue increase or decrease?
d. If the unit price is increased slightly from \$40, will the revenue increase or decrease?

31. Elasticity of Demand The proprietor of the Showplace, a video club, has estimated that the rental price p (in dollars) of prerecorded videocassette tapes is related to the quantity x rented per week by the demand equation

$$x = \frac{2}{3}\sqrt{36 - p^2} \qquad (0 \le p \le 6)$$

Currently, the rental price is \$2/tape.
a. Is the demand elastic or inelastic at this rental price?
b. If the rental price is increased, will the revenue increase or decrease?

32. Elasticity of Demand The demand function for a certain make of exercise bicycle sold exclusively through cable television is

$$p = \sqrt{9 - 0.02x} \qquad (0 \le x \le 450)$$

where p is the unit price in hundreds of dollars and x is the quantity demanded per week. Compute the elasticity of demand and determine the range of prices corresponding to inelastic, unitary, and elastic demand.
Hint: Solve the equation $E(p) = 1$.

33. **Elasticity of Demand** The demand equation for the Sicard wristwatch is given by

$$x = 10\sqrt{\frac{50 - p}{p}} \qquad (0 < p \leq 50)$$

where x (measured in units of a thousand) is the quantity demanded per week and p is the unit price in dollars. Compute the elasticity of demand and determine the range of prices corresponding to inelastic, unitary, and elastic demand.

In Exercises 34 and 35, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

34. If C is a differentiable total cost function, then the marginal average cost function is

$$\overline{C}'(x) = \frac{xC'(x) - C(x)}{x^2}$$

35. If the marginal profit function is positive at $x = a$, then it makes sense to decrease the level of production.

Solutions to Self-Check Exercises 11.4

1. **a.** $R(x) = px$

$$\begin{aligned} &= x(-0.02x + 300) \\ &= -0.02x^2 + 300x \qquad (0 \leq x \leq 15{,}000) \end{aligned}$$

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.02x^2 + 300x \\ &\quad -(0.000003x^3 - 0.04x^2 + 200x + 70{,}000) \\ &= -0.000003x^3 + 0.02x^2 + 100x - 70{,}000 \end{aligned}$$

b. $C'(x) = 0.000009x^2 - 0.08x + 200$

$R'(x) = -0.04x + 300$

$P'(x) = -0.000009x^2 + 0.04x + 100$

c. The average cost function is

$$\begin{aligned} \overline{C}(x) &= \frac{C(x)}{x} \\ &= \frac{0.000003x^3 - 0.04x^2 + 200x + 70{,}000}{x} \\ &= 0.000003x^2 - 0.04x + 200 + \frac{70{,}000}{x} \end{aligned}$$

Therefore, the marginal average cost function is

$$\overline{C}'(x) = 0.000006x - 0.04 - \frac{70{,}000}{x^2}$$

d. Using the results from part (b), we find

$$\begin{aligned} C'(3000) &= 0.000009(3000)^2 - 0.08(3000) + 200 \\ &= 41 \end{aligned}$$

That is, when the level of production is already 3000 VCRs, the actual cost of producing one additional VCR is approximately \$41. Next,

$$R'(3000) = -0.04(3000) + 300 = 180$$

That is, the actual revenue to be realized from selling the 3001st VCR is approximately \$180. Finally,

$$\begin{aligned} P'(3000) &= -0.000009(3000)^2 + 0.04(3000) + 100 \\ &= 139 \end{aligned}$$

That is, the actual profit realized from selling the 3001st VCR is approximately \$139.

2. We first solve the given demand equation for x in terms of p, obtaining

$$\begin{aligned} x = f(p) &= -50p + 15{,}000 \\ f'(p) &= -50 \end{aligned}$$

Therefore,

$$\begin{aligned} E(p) &= -\frac{pf'(p)}{f(p)} = -\frac{p}{-50p + 15{,}000}(-50) \\ &= \frac{p}{300 - p} \qquad (0 \le p < 300) \end{aligned}$$

Next, we compute

$$E(100) = \frac{100}{300 - 100} = \frac{1}{2} < 1$$

and we conclude that demand is inelastic when $p = 100$. Also,

$$E(200) = \frac{200}{300 - 200} = 2 > 1$$

and we see that demand is elastic when $p = 200$.

11.5 Higher-Order Derivatives

Higher-Order Derivatives

The derivative f' of a function f is also a function. As such, the differentiability of f' may be considered. Thus, the function f' has a derivative f'' at a point x in the domain of f' if the limit of the quotient

$$\frac{f'(x + h) - f'(x)}{h}$$

exists as h approaches zero. In other words, it is the derivative of the first derivative.

The function f'' obtained in this manner is called the **second derivative of the function *f***, just as the derivative f' of f is often called the first derivative of f. Continuing in this fashion, we are led to considering the third, fourth, and higher-order derivatives of f whenever they exist. Notations for the first, second, third, and, in general, nth derivatives of a function f at a point x are

$$f'(x),\ f''(x),\ f'''(x),\ \ldots,\ f^{(n)}(x)$$

or

$$D^1f(x),\ D^2f(x),\ D^3f(x),\ \ldots,\ D^nf(x)$$

If f is written in the form $y = f(x)$, then the notations for its derivatives are

$$y', y'', y''', \ldots, y^{(n)}$$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$$

or

$$D^1y, D^2y, D^3y, \ldots, D^ny$$

respectively.

EXAMPLE 1 Find the derivatives of all orders of the polynomial function

$$f(x) = x^5 - 3x^4 + 4x^3 - 2x^2 + x - 8$$

SOLUTION ✔ We have

$$f'(x) = 5x^4 - 12x^3 + 12x^2 - 4x + 1$$

$$f''(x) = \frac{d}{dx}f'(x) = 20x^3 - 36x^2 + 24x - 4$$

$$f'''(x) = \frac{d}{dx}f''(x) = 60x^2 - 72x + 24$$

$$f^{(4)}(x) = \frac{d}{dx}f'''(x) = 120x - 72$$

$$f^{(5)}(x) = \frac{d}{dx}f^{(4)}(x) = 120$$

and, in general,

$$f^{(n)}(x) = 0 \qquad (\text{for } n > 5)$$

■■■■

EXAMPLE 2 Find the third derivative of the function f defined by $y = x^{2/3}$. What is its domain?

SOLUTION ✔ We have

$$y' = \frac{2}{3}x^{-1/3}$$

$$y'' = \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)x^{-4/3} = -\frac{2}{9}x^{-4/3}$$

so the required derivative is

$$y''' = \left(-\frac{2}{9}\right)\left(-\frac{4}{3}\right)x^{-7/3} = \frac{8}{27}x^{-7/3} = \frac{8}{27x^{7/3}}$$

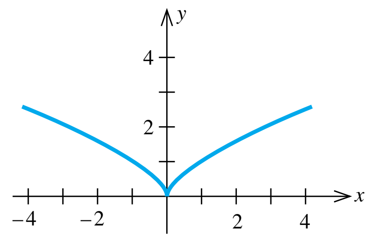


FIGURE 11.13
The graph of the function $y = x^{2/3}$

The common domain of the functions f', f'', and f''' is the set of all real numbers except $x = 0$. The domain of $y = x^{2/3}$ is the set of all real numbers. The graph of the function $y = x^{2/3}$ appears in Figure 11.13. ■■■■

REMARK Always simplify an expression before differentiating it to obtain the next order derivative. ■■■

EXAMPLE 3 Find the second derivative of the function $y = (2x^2 + 3)^{3/2}$.

SOLUTION ✔ We have, using the general power rule,

$$y' = \frac{3}{2}(2x^2 + 3)^{1/2}(4x) = 6x(2x^2 + 3)^{1/2}$$

Next, using the product rule and then the chain rule, we find

$$\begin{aligned} y'' &= (6x) \cdot \frac{d}{dx}(2x^2 + 3)^{1/2} + \left[\frac{d}{dx}(6x)\right](2x^2 + 3)^{1/2} \\ &= (6x)\left(\frac{1}{2}\right)(2x^2 + 3)^{-1/2}(4x) + 6(2x^2 + 3)^{1/2} \\ &= 12x^2(2x^2 + 3)^{-1/2} + 6(2x^2 + 3)^{1/2} \\ &= 6(2x^2 + 3)^{-1/2}[2x^2 + (2x^2 + 3)] \\ &= \frac{6(4x^2 + 3)}{\sqrt{2x^2 + 3}} \end{aligned}$$

■■■■

Applications

Just as the derivative of a function f at a point x measures the rate of change of the function f at that point, the second derivative of f (the derivative of f') measures the rate of change of the derivative f' of the function f. The third derivative of the function f, f''', measures the rate of change of f'', and so on.

In Chapter 12 we will discuss applications involving the geometric interpretation of the second derivative of a function. The following example gives an interpretation of the second derivative in a familiar role.

EXAMPLE 4 Refer to the example on page 621. The distance s (in feet) covered by a maglev moving along a straight track t seconds after starting from rest is given by the function $s = 4t^2$ $(0 \le t \le 10)$. What is the maglev's acceleration at the end of 30 seconds?

SOLUTION ✔ The velocity of the maglev t seconds from rest is given by

$$v = \frac{ds}{dt} = \frac{d}{dt}(4t^2) = 8t$$

The acceleration of the maglev t seconds from rest is given by the rate of change of the velocity of t—that is,

$$a = \frac{d}{dt}v = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2} = \frac{d}{dt}(8t) = 8$$

or 8 feet per second per second, normally abbreviated, 8 ft/sec^2. ■■■■

EXAMPLE 5 A ball is thrown straight up into the air from the roof of a building. The height of the ball as measured from the ground is given by

$$s = -16t^2 + 24t + 120$$

where s is measured in feet and t in seconds. Find the velocity and acceleration of the ball 3 seconds after it is thrown into the air.

SOLUTION ✔ The velocity v and acceleration a of the ball at any time t are given by

$$v = \frac{ds}{dt} = \frac{d}{dt}(-16t^2 + 24t + 120) = -32t + 24$$

and

$$a = \frac{d^2t}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d}{dt}(-32t + 24) = -32$$

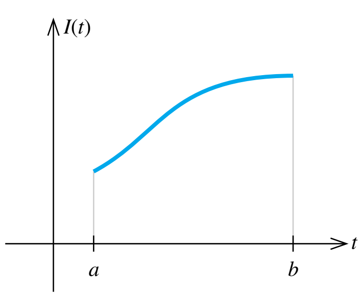
Therefore, the velocity of the ball 3 seconds after it is thrown into the air is

$$v = -32(3) + 24 = -72$$

That is, the ball is falling downward at a speed of 72 ft/sec. The acceleration of the ball is 32 ft/sec^2 downward at any time during the motion. ■■■■

Another interpretation of the second derivative of a function—this time from the field of economics—follows. Suppose the consumer price index (CPI) of an economy between the years a and b is described by the function $I(t)$ $(a \le t \le b)$ (Figure 11.14). Then, the first derivative of I, $I'(t)$, gives the rate of inflation of the economy at any time t. The second derivative of I, $I''(t)$, gives the *rate of change of the inflation rate* at any time t. Thus, when the economist or politician claims that "inflation is slowing," what he or she is saying is that the rate of inflation is decreasing. Mathematically, this is equivalent to noting that the second derivative $I''(t)$ is negative at the time t under consideration. Observe that $I'(t)$ could be positive at a time when $I''(t)$ is negative (see Example 6). Thus, one may not draw the conclusion from the aforementioned quote that prices of goods and services are about to drop!

FIGURE 11.14
The CPI of a certain economy from year a to year b is given by $I(t)$.



EXAMPLE 6

An economy's CPI is described by the function

$$I(t) = -0.2t^3 + 3t^2 + 100 \qquad (0 \le t \le 9)$$

where $t = 0$ corresponds to the year 1992. Compute $I'(6)$ and $I''(6)$ and use these results to show that even though the CPI was rising at the beginning of 1998, "inflation was moderating" at that time.

SOLUTION ✔

We find

$$I'(t) = -0.6t^2 + 6t \qquad \text{and} \qquad I''(t) = -1.2t + 6$$

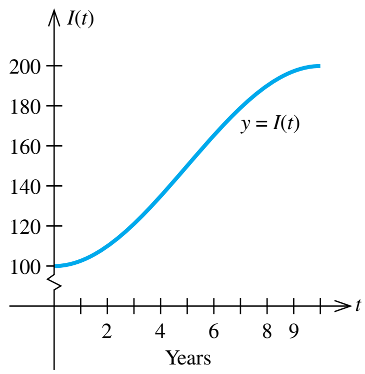
Thus,

$$I'(6) = -0.6(6)^2 + 6(6) = 14.4$$

$$I''(6) = -1.2(6) + 6 = -1.2$$

Our computations reveal that at the beginning of 1998 ($t = 6$), the CPI was increasing at the rate of 14.4 points per year, whereas the rate of the inflation rate was decreasing by 1.2 points per year. Thus, inflation was moderating at that time (Figure 11.15). In Section 12.2, we will see that relief actually began in early 1997. ■■■■

FIGURE 11.15
The CPI of an economy is given by $I(t)$.



SELF-CHECK EXERCISES 11.5

1. Find the third derivative of

$$f(x) = 2x^5 - 3x^3 + x^2 - 6x + 10$$

2. Let

$$f(x) = \frac{1}{1 + x}$$

Find $f'(x)$, $f''(x)$, and $f'''(x)$.

3. A certain species of turtle faces extinction because dealers collect truckloads of turtle eggs to be sold as aphrodisiacs. After severe conservation measures are implemented, it is hoped that the turtle population will grow according to the rule

$$N(t) = 2t^3 + 3t^2 - 4t + 1000 \qquad (0 \le t \le 10)$$

where $N(t)$ denotes the population at the end of year t. Compute $N''(2)$ and $N''(8)$ and interpret your results.

Solutions to Self-Check Exercises 11.5 can be found on page 758.

11.5 Exercises

In Exercises 1–20, find the first and second derivatives of the given function.

1. $f(x) = 4x^2 - 2x + 1$
2. $f(x) = -0.2x^2 + 0.3x + 4$
3. $f(x) = 2x^3 - 3x^2 + 1$
4. $g(x) = -3x^3 + 24x^2 + 6x - 64$
5. $h(t) = t^4 - 2t^3 + 6t^2 - 3t + 10$
6. $f(x) = x^5 - x^4 + x^3 - x^2 + x - 1$
7. $f(x) = (x^2 + 2)^5$
8. $g(t) = t^2(3t + 1)^4$
9. $g(t) = (2t^2 - 1)^2(3t^2)$
10. $h(x) = (x^2 + 1)^2(x - 1)$
11. $f(x) = (2x^2 + 2)^{7/2}$
12. $h(w) = (w^2 + 2w + 4)^{5/2}$
13. $f(x) = x(x^2 + 1)^2$
14. $g(u) = u(2u - 1)^3$
15. $f(x) = \dfrac{x}{2x + 1}$
16. $g(t) = \dfrac{t^2}{t - 1}$
17. $f(s) = \dfrac{s - 1}{s + 1}$
18. $f(u) = \dfrac{u}{u^2 + 1}$
19. $f(u) = \sqrt{4 - 3u}$
20. $f(x) = \sqrt{2x - 1}$

In Exercises 21–28, find the third derivative of the given function.

21. $f(x) = 3x^4 - 4x^3$
22. $f(x) = 3x^5 - 6x^4 + 2x^2 - 8x + 12$
23. $f(x) = \dfrac{1}{x}$
24. $f(x) = \dfrac{2}{x^2}$
25. $g(s) = \sqrt{3s - 2}$
26. $g(t) = \sqrt{2t + 3}$
27. $f(x) = (2x - 3)^4$
28. $g(t) = (\frac{1}{2}t^2 - 1)^5$

29. **Acceleration of a Falling Object** During the construction of an office building, a hammer is accidentally dropped from a height of 256 ft. The distance the hammer falls in t sec is $s = 16t^2$. What is the hammer's velocity when it strikes the ground? What is its acceleration?

30. **Acceleration of a Car** The distance s (in feet) covered by a car t sec after starting from rest is given by

$$s = -t^3 + 8t^2 + 20t \qquad (0 \le t \le 6)$$

Find a general expression for the car's acceleration at any time $t (0 \le t \le 6)$. Show that the car is decelerating $2\frac{2}{3}$ sec after starting from rest.

31. **Crime Rates** The number of major crimes committed in Bronxville between 1988 and 1995 is approximated by the function

$$N(t) = -0.1t^3 + 1.5t^2 + 100 \qquad (0 \le t \le 7)$$

where $N(t)$ denotes the number of crimes committed in year t and $t = 0$ corresponds to the year 1988. Enraged by the dramatic increase in the crime rate, Bronxville's citizens, with the help of the local police, organized "Neighborhood Crime Watch" groups in early 1992 to combat this menace.
a. Verify that the crime rate was increasing from 1988 through 1995.
Hint: Compute $N'(0), N'(1), \ldots, N'(7)$.
b. Show that the Neighborhood Crime Watch program was working by computing $N''(4)$, $N''(5)$, $N''(6)$, and $N''(7)$.

32. **GDP of a Developing Country** A developing country's gross domestic product (GDP) from 1992 to 2000 is approximated by the function

$$G(t) = -0.2t^3 + 2.4t^2 + 60 \qquad (0 \le t \le 8)$$

where $G(t)$ is measured in billions of dollars and $t = 0$ corresponds to the year 1992.
a. Compute $G'(0), G'(1), \ldots, G'(8)$.
b. Compute $G''(0), G''(1), \ldots, G''(8)$.
c. Using the results obtained in parts (a) and (b), show that after a spectacular growth rate in the early years, the growth of the GDP cooled off.

33. **Test Flight of a VTOL** In a test flight of the McCord Terrier, McCord Aviation's experimental VTOL (vertical takeoff and landing) aircraft, it was determined that t sec after lift-off, when the craft was operated in the

(continued on p. 758)

Using Technology

FINDING THE SECOND DERIVATIVE OF A FUNCTION AT A GIVEN POINT

Some graphing utilities have the capability of numerically computing the second derivative of a function at a point. If your graphing utility has this capability, use it to work through the examples and exercises of this section.

EXAMPLE 1 Use the (second) numerical derivative operation of a graphing utility to find the second derivative of $f(x) = \sqrt{x}$ when $x = 4$.

SOLUTION ✔ Using the (second) numerical derivative operation of a graphing utility, we find

$$f''(4) = \text{der2}\ (x\text{^}.5, x, 4) = -0.03125$$

EXAMPLE 2 The anticipated rise in Alzheimer's patients in the United States is given by

$$f(t) = -0.02765t^4 + 0.3346t^3 - 1.1261t^2 + 1.7575t + 3.7745 \qquad (0 \le t \le 6)$$

where $f(t)$ is measured in millions and t is measured in decades, with $t = 0$ corresponding to the beginning of 1990.

a. How fast is the number of Alzheimer's patients in the United States anticipated to be changing at the beginning of 2030?
b. How fast is the rate of change of the number of Alzheimer's patients in the United States anticipated to be changing at the beginning of 2030?
c. Plot the graph of f in the viewing rectangle $[0, 7] \times [0, 12]$.

Source: Alzheimer's Association

SOLUTION ✔

a. Using the numerical derivative operation of a graphing utility, we find that the number of Alzheimer's patients at the beginning of 2030 can be anticipated to be changing at the rate of

$$f'(4) = 1.7311$$

That is, the number is increasing at the rate of approximately 1.7 million patients per decade.

b. Using the (second) numerical derivative operation of a graphing utility, we find that

$$f''(4) = 0.4694$$

That is, the rate of change of the number of Alzheimer's patients is increasing at the rate of approximately 0.5 million patients per decade per decade.

c. The graph is shown in Figure T1.

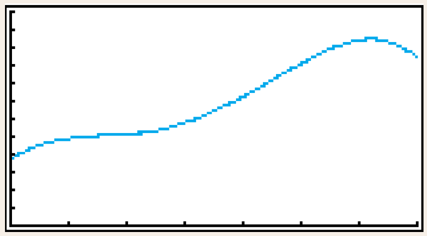


FIGURE T1
The graph of f in the viewing window $[0, 7] \times [0, 12]$

Exercises

In Exercises 1–8, find the value of the second derivative of f at the given value of x. Express your answer correct to four decimal places.

1. $f(x) = 2x^3 - 3x^2 + 1;\ x = -1$
2. $f(x) = 2.5x^5 - 3x^3 + 1.5x + 4;\ x = 2.1$
3. $f(x) = 2.1x^{3.1} - 4.2x^{1.7} + 4.2;\ x = 1.4$
4. $f(x) = 1.7x^{4.2} - 3.2x^{1.3} + 4.2x - 3.2;\ x = 2.2$
5. $f(x) = \dfrac{x^2 + 2x - 5}{x^3 + 1};\ x = 2.1$
6. $f(x) = \dfrac{x^3 + x + 2}{2x^2 - 5x + 4};\ x = 1.2$
7. $f(x) = \dfrac{x^{1/2} + 2x^{3/2} + 1}{2x^{1/2} + 3};\ x = 0.5$
8. $f(x) = \dfrac{\sqrt{x} - 1}{2x + \sqrt{x} + 4};\ x = 2.3$
9. **Rate of Bank Failures** The Federal Deposit Insurance Corporation (FDIC) estimates that the rate at which banks were failing between 1982 and 1994 is given by

$$f(t) = -0.063447t^4 - 1.953283t^3 + 14.632576t^2 - 6.684704t + 47.458874 \qquad (0 \le t \le 12)$$

where $f(t)$ is measured in the number of banks per year and t is measured in years, with $t = 0$ corresponding to the beginning of 1982. Compute $f''(6)$ and interpret your results.
Source: Federal Deposit Insurance Corporation

10. **Multimedia Sales** According to the Electronic Industries Association, sales in the multimedia market (hardware and software) are expected to be

$$S(t) = -0.0094t^4 + 0.1204t^3 - 0.0868t^2 + 0.0195t + 3.3325 \qquad (0 \le t \le 10)$$

where $S(t)$ is measured in billions of dollars and t is measured in years, with $t = 0$ corresponding to 1990. Compute $S''(7)$ and interpret your results.
Source: Electronics Industries Association

vertical takeoff mode, its altitude (in feet) was

$$h(t) = \frac{1}{16}t^4 - t^3 + 4t^2 \qquad (0 \le t \le 8)$$

a. Find an expression for the craft's velocity at time t.
b. Find the craft's velocity when $t = 0$ (the initial velocity), $t = 4$, and $t = 8$.
c. Find an expression for the craft's acceleration at time t.
d. Find the craft's acceleration when $t = 0$, 4, and 8.
e. Find the craft's height when $t = 0$, 4, and 8.

34. U.S. Census According to the U.S. Census Bureau, the number of Americans aged 45 to 54 will be approximately

$$N(t) = -0.00233t^4 + 0.00633t^3 - 0.05417t^2 + 1.3467t + 25$$

million people in year t, where $t = 0$ corresponds to the beginning of 1990. Compute $N'(10)$ and $N''(10)$ and interpret your results.
Source: U.S. Census Bureau

35. Air Purification During testing of a certain brand of air purifier, it was determined that the amount of smoke remaining t min after the start of the test was

$$A(t) = -0.00006t^5 + 0.00468t^4 - 0.1316t^3 + 1.915t^2 - 17.63t + 100$$

percent of the original amount. Compute $A'(10)$ and $A''(10)$ and interpret your results.
Source: Consumer Reports

In Exercises 36–39, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

36. If the second derivative of f exists at $x = a$, then $f''(a) = [f'(a)]^2$.

37. If $h = fg$ where f and g have second-order derivatives, then

$$h''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

38. If $f(x)$ is a polynomial function of degree n, then $f^{(n+1)}(x) = 0$.

39. Suppose $P(t)$ represents the population of bacteria at time t and suppose $P'(t) > 0$ and $P''(t) < 0$; then the population is increasing at time t but at a decreasing rate.

40. Let f be the function defined by the rule $f(x) = x^{7/3}$. Show that f has first- and second-order derivatives at all points x, in particular at $x = 0$. Show also that the third derivative of f does *not* exist at $x = 0$.

41. Construct a function f that has derivatives of order up through and including n at a point a but fails to have the $(n + 1)$st derivative there.
Hint: See Exercise 40.

42. Show that a polynomial function has derivatives of all orders.
Hint: Let $P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ be a polynomial of degree n, where n is a positive integer and $a_0, a_1, \ldots, a_n$ are constants with $a_0 \neq 0$. Compute $P'(x), P''(x), \ldots$.

Solutions to Self-Check Exercises 11.5

1. $f'(x) = 10x^4 - 9x^2 + 2x - 6$
$f''(x) = 40x^3 - 18x + 2$
$f'''(x) = 120x^2 - 18$

2. We write $f(x) = (1 + x)^{-1}$ and use the general power rule, obtaining

$$f'(x) = (-1)(1 + x)^{-2}\frac{d}{dx}(1 + x) = -(1 + x)^{-2}(1)$$
$$= -(1 + x)^{-2} = -\frac{1}{(1 + x)^2}$$

Continuing, we find

$$\begin{aligned} f''(x) &= -(-2)(1+x)^{-3} \\ &= 2(1+x)^{-3} = \frac{2}{(1+x)^3} \\ f'''(x) &= 2(-3)(1+x)^{-4} \\ &= -6(1+x)^{-4} \\ &= -\frac{6}{(1+x)^4} \end{aligned}$$

3. $N'(t) = 6t^2 + 6t - 4$

$N''(t) = 12t + 6 = 6(2t + 1)$

Therefore, $N''(2) = 30$ and $N''(8) = 102$. The results of our computations reveal that at the end of year 2, the *rate* of growth of the turtle population is increasing at the rate of 30 turtles/year/year. At the end of year 8, the rate is increasing at the rate of 102 turtles/year/year. Clearly, the conservation measures are paying off handsomely.

11.6 Implicit Differentiation and Related Rates

Differentiating Implicitly

Up to now we have dealt with functions expressed in the form $y = f(x)$; that is, the dependent variable y is expressed *explicitly* in terms of the independent variable x. However, not all functions are expressed in this form. Consider, for example, the equation

$$x^2y + y - x^2 + 1 = 0 \tag{8}$$

This equation does express y *implicitly* as a function of x. In fact, solving (8) for y in terms of x, we obtain

$$(x^2 + 1)y = x^2 - 1 \qquad \text{(Implicit equation)}$$

$$y = f(x) = \frac{x^2 - 1}{x^2 + 1} \qquad \text{(Explicit equation)}$$

which gives an explicit representation of f.

Next, consider the equation

$$y^4 - y^3 - y + 2x^3 - x = 8$$

When certain restrictions are placed on x and y, this equation defines y as a function of x. But in this instance, we would be hard pressed to find y explicitly in terms of x. The following question arises naturally: How does one go about computing dy/dx in this case?

As it turns out, thanks to the chain rule, a method *does* exist for computing the derivative of a function directly from the implicit equation defining the function. This method is called **implicit differentiation** and is demonstrated in the next several examples.

EXAMPLE 1 Find $\frac{dy}{dx}$ given the equation $y^2 = x$.

SOLUTION ✔ Differentiating both sides of the equation with respect to x, we obtain

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

To carry out the differentiation of the term $\frac{d}{dx}y^2$, we note that y is a function of x. Writing $y = f(x)$ to remind us of this fact, we find that

$$\begin{aligned}\frac{d}{dx}(y^2) &= \frac{d}{dx}[f(x)]^2 && \text{[Writing } y = f(x)\text{]}\\ &= 2f(x)f'(x) && \text{(Using the chain rule)}\\ &= 2y\frac{dy}{dx} && \text{[Returning to using } y \text{ instead of } f(x)\text{]}\end{aligned}$$

Therefore, the equation

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

is equivalent to

$$2y\frac{dy}{dx} = 1$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{2y}$$

■■■■

Before considering other examples, let us summarize the important steps involved in implicit differentiation. (Here we assume that dy/dx exists.)

Finding $\frac{dy}{dx}$ by Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*. (Make sure that the derivative of any term involving y includes the factor dy/dx.)
2. Solve the resulting equation for dy/dx in terms of x and y.

EXAMPLE 2 Find dy/dx given the equation

$$y^3 - y + 2x^3 - x = 8$$

SOLUTION ✔ Differentiating both sides of the given equation with respect to x, we obtain

$$\frac{d}{dx}(y^3 - y + 2x^3 - x) = \frac{d}{dx}(8)$$

$$\frac{d}{dx}(y^3) - \frac{d}{dx}(y) + \frac{d}{dx}(2x^3) - \frac{d}{dx}(x) = 0$$

Group Discussion
Refer to Example 2. Suppose we think of the equation $y^3 - y + 2x^3 - x = 8$ as defining x implicitly as a function of y. Find dx/dy and justify your method of solution.

Now, recalling that y is a function of x, we apply the chain rule to the first two terms on the left. Thus,

$$3y^2\frac{dy}{dx} - \frac{dy}{dx} + 6x^2 - 1 = 0$$
$$(3y^2 - 1)\frac{dy}{dx} = 1 - 6x^2$$
$$\frac{dy}{dx} = \frac{1 - 6x^2}{3y^2 - 1}$$

EXAMPLE 3 Consider the equation $x^2 + y^2 = 4$.

a. Find dy/dx by implicit differentiation.
b. Find the slope of the tangent line to the graph of the function $y = f(x)$ at the point $(1, \sqrt{3})$.
c. Find an equation of the tangent line of part (b).

SOLUTION **a.** Differentiating both sides of the equation with respect to x, we obtain

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(4)$$
$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{x}{y} \qquad (y \neq 0)$$

b. The slope of the tangent line to the graph of the function at the point $(1, \sqrt{3})$ is given by

$$\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = -\left.\frac{x}{y}\right|_{(1,\sqrt{3})} = -\frac{1}{\sqrt{3}}$$

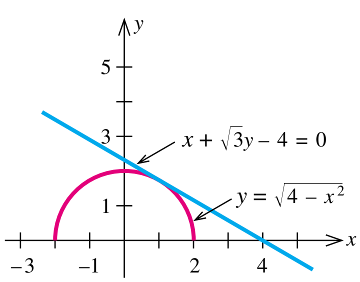
(*Note:* This notation is read "dy/dx evaluated at the point $(1, \sqrt{3})$.")

c. An equation of the tangent line in question is found by using the point-slope form of the equation of a line with the slope $m = -1/\sqrt{3}$ and the point $(1, \sqrt{3})$. Thus,

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x - 1)$$
$$\sqrt{3}y - 3 = -x + 1$$
$$x + \sqrt{3}y - 4 = 0$$

A sketch of this tangent line is shown in Figure 11.16.

FIGURE 11.16
The line $x + \sqrt{3}y - 4 = 0$ is tangent to the graph of the function $y = f(x)$.



We can also solve the equation $x^2 + y^2 = 4$ explicitly for y in terms of x. If we do this, we obtain

$$y = \pm\sqrt{4 - x^2}$$

From this, we see that the equation $x^2 + y^2 = 4$ defines the two functions

$$y = f(x) = \sqrt{4 - x^2}$$
$$y = g(x) = -\sqrt{4 - x^2}$$

Since the point $(1, \sqrt{3})$ does not lie on the graph of $y = g(x)$, we conclude that

$$y = f(x) = \sqrt{4 - x^2}$$

is the required function. The graph of f is the upper semicircle shown in Figure 11.16.

Group Discussion

Refer to Example 3. Yet another function defined implicitly by the equation $x^2 + y^2 = 4$ is the function

$$y = h(x) = \begin{cases} \sqrt{4 - x^2} & \text{if } -2 \le x < 0 \\ -\sqrt{4 - x^2} & \text{if } \ \ 0 \le x \le 2 \end{cases}$$

1. Sketch the graph of h.
2. Show that $h'(x) = -x/y$.
3. Find an equation of the tangent line to the graph of h at the point $(1, -\sqrt{3})$.

To find dy/dx at a *specific point* (a, b), differentiate the given equation implicitly with respect to x and then replace x and y by a and b, respectively, *before* solving the equation for dy/dx. This often simplifies the amount of algebra involved.

EXAMPLE 4 Find dy/dx given that x and y are related by the equation

$$x^2y^3 + 6x^2 = y + 12$$

and that $y = 2$ when $x = 1$.

SOLUTION Differentiating both sides of the given equation with respect to x, we obtain

$$\frac{d}{dx}(x^2y^3) + \frac{d}{dx}(6x^2) = \frac{d}{dx}(y) + \frac{d}{dx}(12)$$

$$x^2 \cdot \frac{d}{dx}(y^3) + y^3 \cdot \frac{d}{dx}(x^2) + 12x = \frac{dy}{dx} \quad \text{[Using the product rule on } \frac{d}{dx}(x^2y^3)\text{]}$$

$$3x^2y^2\frac{dy}{dx} + 2xy^3 + 12x = \frac{dy}{dx}$$

Substituting $x = 1$ and $y = 2$ into this equation gives

$$3(1)^2(2)^2\frac{dy}{dx} + 2(1)(2)^3 + 12(1) = \frac{dy}{dx}$$
$$12\frac{dy}{dx} + 16 + 12 = \frac{dy}{dx}$$

and, solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = -\frac{28}{11}$$

Note that it is not necessary to find an explicit expression for dy/dx. ■■■■

REMARK In Examples 3 and 4, you can verify that the points at which we evaluated dy/dx actually lie on the curve in question by showing that the coordinates of the points satisfy the given equations. ■■■

EXAMPLE 5 Find dy/dx given that x and y are related by the equation

$$\sqrt{x^2 + y^2} - x^2 = 5$$

SOLUTION ✔ Differentiating both sides of the given equation with respect to x, we obtain

$$\frac{d}{dx}(x^2 + y^2)^{1/2} - \frac{d}{dx}(x^2) = \frac{d}{dx}(5) \qquad \left[\text{Writing } \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}\right]$$

$$\frac{1}{2}(x^2 + y^2)^{-1/2}\frac{d}{dx}(x^2 + y^2) - 2x = 0 \qquad \text{(Using the general power rule on the first term)}$$

$$\frac{1}{2}(x^2 + y^2)^{-1/2}\left(2x + 2y\frac{dy}{dx}\right) - 2x = 0$$

$$2x + 2y\frac{dy}{dx} = 4x(x^2 + y^2)^{1/2} \qquad \left[\text{Transposing } 2x \text{ and multiplying both sides by } 2(x^2 + y^2)^{1/2}\right]$$

$$2y\frac{dy}{dx} = 4x(x^2 + y^2)^{1/2} - 2x$$

$$\frac{dy}{dx} = \frac{2x\sqrt{x^2 + y^2} - x}{y}$$

■■■■

Related Rates

Implicit differentiation is a useful technique for solving a class of problems known as **related rates** problems. For example, suppose x and y are each functions of a third variable t. Here, x might denote the mortgage rate and y the number of single-family homes sold at any time t. Further, suppose we have an equation that gives the relationship between x and y (the number of houses sold y is related to the mortgage rate x). Differentiating both sides of this equation implicitly with respect to t, we obtain an equation that gives a relationship between dx/dt and dy/dt. In the context of our example, this equation gives us a relationship between the rate of change of the mortgage rate and the rate of change of the number of houses sold, as a function of time. Thus, knowing

$$\frac{dx}{dt} \qquad \text{(How fast the mortgage rate is changing at time } t\text{)}$$

we can determine

$$\frac{dy}{dt} \qquad \text{(How fast the sale of houses is changing at that instant of time)}$$

EXAMPLE 6 A study prepared for the National Association of Realtors estimates that the number of housing starts in the Southwest, $N(t)$ (in units of a million), over the next 5 years is related to the mortgage rate $r(t)$ (percent per year) by the equation

$$9N^2 + r = 36$$

What is the rate of change of the number of housing starts with respect to time when the mortgage rate is 11% per year and is increasing at the rate of 1.5% per year?

SOLUTION ✓ We are given that

$$r = 11 \qquad \text{and} \qquad \frac{dr}{dt} = 1.5$$

at a certain instant of time, and we are required to find dN/dt. First, by substituting $r = 11$ into the given equation, we find

$$9N^2 + 11 = 36$$
$$N^2 = \frac{25}{9}$$

or $N = 5/3$ (we reject the negative root). Next, differentiating the given equation implicitly on both sides with respect to t, we obtain

$$\frac{d}{dt}(9N^2) + \frac{d}{dt}(r) = \frac{d}{dt}(36)$$
$$18N\frac{dN}{dt} + \frac{dr}{dt} = 0 \qquad \text{(Use the chain rule on the first term.)}$$

Then, substituting $N = 5/3$ and $dr/dt = 1.5$ into this equation gives

$$18\left(\frac{5}{3}\right)\frac{dN}{dt} + 1.5 = 0$$

Solving this equation for dN/dt then gives

$$\frac{dN}{dt} = -\frac{1.5}{30} \approx -0.05$$

Thus, at the instant of time under consideration, the number of housing starts is decreasing at the rate of 50,000 units per year. ■■■■

EXAMPLE 7 A major audiotape manufacturer is willing to make x thousand ten packs of metal alloy audiocassette tapes per week available in the marketplace when the wholesale price is $\$p$ per ten pack. It is known that the relationship between x and p is governed by the supply equation

$$x^2 - 3xp + p^2 = 5$$

How fast is the supply of tapes changing when the price per ten pack is \$11, the quantity supplied is 4000 ten packs, and the wholesale price per ten pack is increasing at the rate of 10 cents per ten pack per week?

SOLUTION ✔ We are given that

$$p = 11, \qquad x = 4, \qquad \frac{dp}{dt} = 0.1$$

at a certain instant of time, and we are required to find dx/dt. Differentiating the given equation on both sides with respect to t, we obtain

$$\frac{d}{dt}(x^2) - \frac{d}{dt}(3xp) + \frac{d}{dt}(p^2) = \frac{d}{dt}(5)$$

$$2x\frac{dx}{dt} - 3\left(p\frac{dx}{dt} + x\frac{dp}{dt}\right) + 2p\frac{dp}{dt} = 0$$

(Use the product rule on the second term.)

Substituting the given values of p, x, and dp/dt into the last equation, we have

$$2(4)\frac{dx}{dt} - 3\left[(11)\frac{dx}{dt} + 4(0.1)\right] + 2(11)(0.1) = 0$$

$$8\frac{dx}{dt} - 33\frac{dx}{dt} - 1.2 + 2.2 = 0$$

$$25\frac{dx}{dt} = 1$$

$$\frac{dx}{dt} = 0.04$$

Thus, at the instant of time under consideration the supply of ten pack audiocassettes is increasing at the rate of (0.04)(1000), or 40, ten packs per week. ■■■■

In certain related problems, we need to formulate the problem mathematically before analyzing it. The following guidelines can be used to help solve problems of this type.

Solving Related Rates Problems

1. Assign a variable to each quantity. Draw a diagram if needed.
2. Write the *given* values of the variables and their rates of change with respect to t.
3. Find an equation giving the relationship between the variables.
4. Differentiate both sides of this equation implicitly with respect to t.
5. Replace the variables and their derivatives by the numerical data found in step 2 and solve the equation for the required rate of change.

EXAMPLE 8 At a distance of 4000 feet from the launch site, a spectator is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 feet/second when it is at an altitude of 3000 feet, how fast is the distance between the rocket and the spectator changing at that instant?

SOLUTION ✓

Step 1 Let

$$y = \text{the altitude of the rocket}$$
$$x = \text{the distance between the rocket and the spectator}$$

at any time t (Figure 11.17).

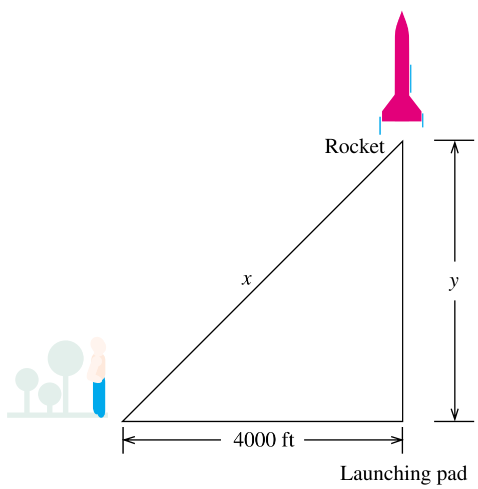


FIGURE 11.17
The rate at which x is changing with respect to time is related to the rate of change of y with respect to time.

Step 2 We are given that at a certain instant of time

$$y = 3000 \qquad \text{and} \qquad \frac{dy}{dt} = 600$$

and are asked to find dx/dt at that instant.

Step 3 Applying the Pythagorean theorem to the right triangle in Figure 11.17, we find that

$$x^2 = y^2 + 4000^2$$

Therefore, when $y = 3000$,

$$x = \sqrt{3000^2 + 4000^2} = 5000$$

Step 4 Next, we differentiate the equation $x^2 = y^2 + 4000^2$ with respect to t, obtaining

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

(Remember, both x and y are functions of t.)

Step 5 Substituting $x = 5000$, $y = 3000$, and $dy/dt = 600$, we find

$$2(5000)\frac{dx}{dt} = 2(3000)(600)$$

$$\frac{dx}{dt} = 360$$

Therefore, the distance between the rocket and the spectator is changing at a rate of 360 feet/second. ■■■■

Be sure that you do *not* replace the variables in the equation found in step 3 by their numerical values before differentiating the equation.

SELF-CHECK EXERCISES 11.6

1. Given the equation $x^3 + 3xy + y^3 = 4$, find dy/dx by implicit differentiation.
2. Find an equation of the tangent line to the graph of $16x^2 + 9y^2 = 144$ at the point $\left(2, -\frac{4\sqrt{5}}{3}\right)$.

Solutions to Self-Check Exercises 11.6 can be found on page 770.

11.6 Exercises

In Exercises 1–8, find the derivative *dy/dx* (a) by solving each of the given implicit equations for *y* explicitly in terms of *x* and (b) by differentiating each of the given equations implicitly. Show that, in each case, the results are equivalent.

1. $x + 2y = 5$
2. $3x + 4y = 6$
3. $xy = 1$
4. $xy - y - 1 = 0$
5. $x^3 - x^2 - xy = 4$
6. $x^2y - x^2 + y - 1 = 0$
7. $\frac{x}{y} - x^2 = 1$
8. $\frac{y}{x} - 2x^3 = 4$

In Exercises 9–30, find *dy/dx* by implicit differentiation.

9. $x^2 + y^2 = 16$
10. $2x^2 + y^2 = 16$
11. $x^2 - 2y^2 = 16$
12. $x^3 + y^3 + y - 4 = 0$
13. $x^2 - 2xy = 6$
14. $x^2 + 5xy + y^2 = 10$
15. $x^2y^2 - xy = 8$
16. $x^2y^3 - 2xy^2 = 5$
17. $x^{1/2} + y^{1/2} = 1$
18. $x^{1/3} + y^{1/3} = 1$
19. $\sqrt{x + y} = x$
20. $(2x + 3y)^{1/3} = x^2$
21. $\frac{1}{x^2} + \frac{1}{y^2} = 1$
22. $\frac{1}{x^3} + \frac{1}{y^3} = 5$
23. $\sqrt{xy} = x + y$
24. $\sqrt{xy} = 2x + y^2$
25. $\frac{x + y}{x - y} = 3x$
26. $\frac{x - y}{2x + 3y} = 2x$
27. $xy^{3/2} = x^2 + y^2$
28. $x^2y^{1/2} = x + 2y^3$
29. $(x + y)^3 + x^3 + y^3 = 0$
30. $(x + y^2)^{10} = x^2 + 25$

In Exercises 31–34, find an equation of the tangent line to the graph of the function f defined by the given equation at the indicated point.

31. $4x^2 + 9y^2 = 36$; $(0, 2)$

32. $y^2 - x^2 = 16$; $(2, 2\sqrt{5})$

33. $x^2y^3 - y^2 + xy - 1 = 0$; $(1, 1)$

34. $(x - y - 1)^3 = x$; $(1, -1)$

In Exercises 35–38, find the second derivative d^2y/dx^2 of each of the functions defined implicitly by the given equation.

35. $xy = 1$

36. $x^3 + y^3 = 28$

37. $y^2 - xy = 8$

38. $x^{1/3} + y^{1/3} = 1$

39. The volume of a right-circular cylinder of radius r and height h is $V = \pi r^2 h$. Suppose the radius and height of the cylinder are changing with respect to time t.
a. Find a relationship between dV/dt, dr/dt, and dh/dt.
b. At a certain instant of time, the radius and height of the cylinder are 2 and 6 in. and are increasing at the rate of 0.1 and 0.3 in./sec, respectively. How fast is the volume of the cylinder increasing?

40. A car leaves an intersection traveling west. Its position 4 seconds later is 20 ft from the intersection. At the same time, another car leaves the same intersection heading north so that its position 4 sec later is 28 ft from the intersection. If the speed of the cars at that instant of time is 9 ft/sec and 11 ft/sec, respectively, find the rate at which the distance between the two cars is changing.

41. PRICE-DEMAND Suppose the quantity demanded weekly of the Super Titan radial tires is related to its unit price by the equation

$$p + x^2 = 144$$

where p is measured in dollars and x is measured in units of a thousand. How fast is the quantity demanded changing when $x = 9$, $p = 63$, and the price per tire is increasing at the rate of \$2/week?

42. PRICE-SUPPLY Suppose the quantity x of Super Titan radial tires made available per week in the marketplace by the Titan Tire Company is related to the unit selling price by the equation

$$p - \frac{1}{2}x^2 = 48$$

where x is measured in units of a thousand and p is in dollars. How fast is the weekly supply of Super Titan radial tires being introduced into the marketplace when $x = 6$, $p = 66$, and the price per tire is decreasing at the rate of \$3/week?

43. PRICE-DEMAND The demand equation for a certain brand of metal alloy audiocassette tape is

$$100x^2 + 9p^2 = 3600$$

where x represents the number (in thousands) of ten packs demanded per week when the unit price is \$$p$. How fast is the quantity demanded increasing when the unit price per ten pack is \$14 and the selling price is dropping at the rate of \$.15 per ten pack per week?
Hint: To find the value of x when $p = 14$, solve the equation $100x^2 + 9p^2 = 3600$ for x when $p = 14$.

44. EFFECT OF PRICE ON SUPPLY Suppose the wholesale price of a certain brand of medium-size eggs p (in dollars per carton) is related to the weekly supply x (in thousands of cartons) by the equation

$$625p^2 - x^2 = 100$$

If 25,000 cartons of eggs are available at the beginning of a certain week and the price is falling at the rate of 2 cents/carton/week, at what rate is the supply falling?
Hint: To find the value of p when $x = 25$, solve the supply equation for p when $x = 25$.

45. SUPPLY-DEMAND Refer to Exercise 44. If 25,000 cartons of eggs are available at the beginning of a certain week and the supply is falling at the rate of 1000 cartons/week, at what rate is the wholesale price changing?

46. ELASTICITY OF DEMAND The demand function for a certain make of cartridge typewriter ribbon is

$$p = -0.01x^2 - 0.1x + 6$$

where p is the unit price in dollars and x is the quantity demanded each week, measured in units of a thousand. Compute the elasticity of demand and determine whether the demand is inelastic, unitary, or elastic when $x = 10$.

47. ELASTICITY OF DEMAND The demand function for a certain brand of compact disc is

$$p = -0.01x^2 - 0.2x + 8$$

where p is the wholesale unit price in dollars and x is the quantity demanded each week, measured in units of a thousand. Compute the elasticity of demand and determine whether the demand is inelastic, unitary, or elastic when $x = 15$.

48. The volume V of a cube with sides of length x in. is changing with respect to time. At a certain instant of time, the sides of the cube are 5 in. long and increasing at the rate of 0.1 in./sec. How fast is the volume of the cube changing at that instant of time?

49. Oil Spills In calm waters, oil spilling from the ruptured hull of a grounded tanker spreads in all directions. If the area polluted is a circle and its radius is increasing at a rate of 2 ft/sec, determine how fast the area is increasing when the radius of the circle is 40 ft.

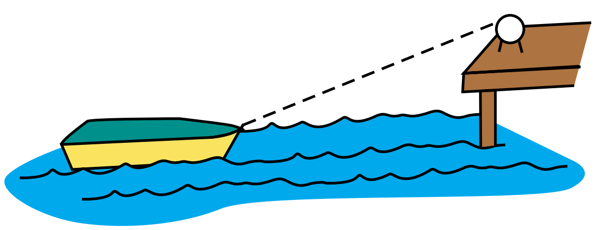
50. Two ships leave the same port at noon. Ship A sails north at 15 mph, and ship B sails east at 12 mph. How fast is the distance between them changing at 1 P.M.?

51. A car leaves an intersection traveling east. Its position t sec later is given by $x = t^2 + t$ ft. At the same time, another car leaves the same intersection heading north, traveling $y = t^2 + 3t$ ft in t sec. Find the rate at which the distance between the two cars will be changing 5 sec later.

52. At a distance of 50 ft from the pad, a man observes a helicopter taking off from a heliport. If the helicopter lifts off vertically and is rising at a speed of 44 ft/sec when it is at an altitude of 120 ft, how fast is the distance between the helicopter and the man changing at that instant?

53. A spectator watches a rowing race from the edge of a river bank. The lead boat is moving in a straight line that is 120 ft from the river bank. If the boat is moving at a constant speed of 20 ft/sec, how fast is the boat moving away from the spectator when it is 50 ft past her?

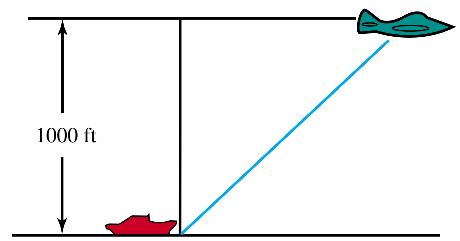
54. A boat is pulled toward a dock by means of a rope which is wound on a drum that is located 4 ft above the bow of the boat. If the rope is being pulled in at the rate of 3 ft/sec, how fast is the boat approaching the dock when it is 25 ft from the dock?



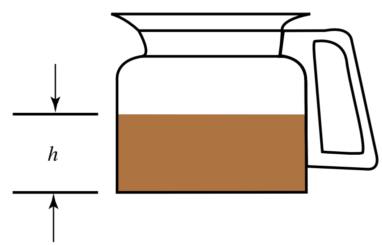
55. Assume that a snowball is in the shape of a sphere. If the snowball melts at a rate that is proportional to its surface area, show that its radius decreases at a constant rate.
Hint: Its volume is $V = (4/3)\pi r^3$, and its surface area is $S = 4\pi r^2$.

56. Blowing Soap Bubbles Carlos is blowing air into a soap bubble at the rate of 8 cm^3/sec. Assuming that the bubble is spherical, how fast is its radius changing at the instant of time when the radius is 10 cm? How fast is the surface area of the bubble changing at that instant of time?

57. Coast Guard Patrol Search Mission The pilot of a Coast Guard patrol aircraft on a search mission had just spotted a disabled fishing trawler and decided to go in for a closer look. Flying at a constant altitude of 1000 ft and at a steady speed of 264 ft/sec, the aircraft passed directly over the trawler. How fast was the aircraft receding from the trawler when it was 1500 ft from it?

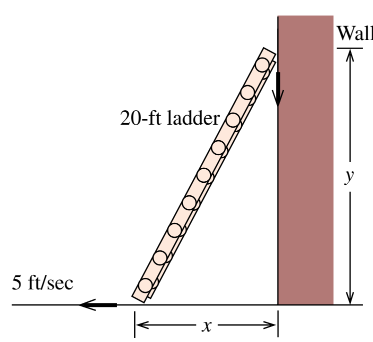


58. A coffee pot in the form of a circular cylinder of radius 4 in. is being filled with water flowing at a constant rate. If the water level is rising at the rate of 0.4 in./sec, what is the rate at which water is flowing into the coffee pot?



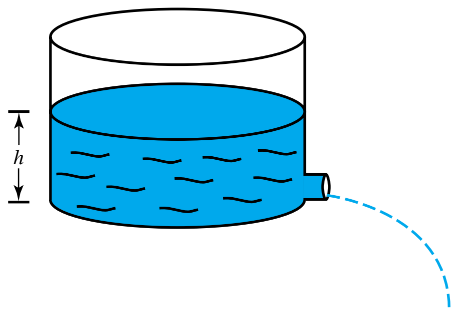
59. A 6-ft tall man is walking away from a street light 18 ft high at a speed of 6 ft/sec. How fast is the tip of his shadow moving along the ground?

60. A 20-ft ladder leaning against a wall begins to slide. How fast is the top of the ladder sliding down the wall at the instant of time when the bottom of the ladder is 12 ft from the wall and sliding away from the wall at the rate of 5 ft/sec?
Hint: Refer to the adjacent figure. By the Pythagorean theorem, $x^2 + y^2 = 400$. Find dy/dt when $x = 12$ and $dx/dt = 5$.



61. The base of a 13-ft ladder leaning against a wall begins to slide away from the wall. At the instant of time when the base is 12 ft from the wall, the base is moving at the rate of 8 ft/sec. How fast is the top of the ladder sliding down the wall at that instant of time?
Hint: Refer to the hint in Problem 60.

62. Water flows from a tank of constant cross-sectional area 50 ft^2 through an orifice of constant cross-sectional area 1.4 ft^2 located at the bottom of the tank (see the figure).



Initially, the height of the water in the tank was 20 ft and its height t sec later is given by the equation

$$2\sqrt{h} + \frac{1}{25}t - 2\sqrt{20} = 0 \qquad (0 \leq t \leq 50\sqrt{20})$$

How fast was the height of the water decreasing when its height was 8 ft?

In Exercises 63 and 64, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

63. If f and g are differentiable and $f(x)g(y) = 0$, then

$$\frac{dy}{dx} = -\frac{f'(x)g(y)}{f(x)g'(y)} \qquad (f(x) \neq 0 \quad \text{and} \quad g'(y) \neq 0)$$

64. If f and g are differentiable and $f(x) + g(y) = 0$, then

$$\frac{dy}{dx} = -\frac{f'(x)}{g'(y)}$$

Solutions to Self-Check Exercises 11.6

1. Differentiating both sides of the equation with respect to x, we have

$$3x^2 + 3y + 3xy' + 3y^2y' = 0$$
$$(x^2 + y) + (x + y^2)y' = 0$$
$$y' = -\frac{x^2 + y}{x + y^2}$$

2. To find the slope of the tangent line to the graph of the function at any point, we differentiate the equation implicitly with respect to x, obtaining

$$32x + 18yy' = 0$$
$$y' = -\frac{16x}{9y}$$

In particular, the slope of the tangent line at $\left(2, -\frac{4\sqrt{5}}{3}\right)$ is

$$m = -\frac{16(2)}{9\left(-\frac{4\sqrt{5}}{3}\right)} = \frac{8}{3\sqrt{5}}$$

Using the point-slope form of the equation of a line, we find

$$y - \left(-\frac{4\sqrt{5}}{3}\right) = \frac{8}{3\sqrt{5}}(x - 2)$$
$$y = \frac{8\sqrt{5}}{15}x - \frac{36\sqrt{5}}{15} = \frac{8\sqrt{5}}{15}x - \frac{12\sqrt{5}}{5}$$

11.7 Differentials

The Millers are planning to buy a house in the near future and estimate that they will need a 30-year fixed-rate mortgage for $120,000. If the interest rate increases from the present rate of 9% per year to 9.4% per year between now and the time the Millers decide to secure the loan, approximately how much more per month will their mortgage be? (You will be asked to answer this question in Exercise 44, page 781.)

Questions such as this, in which one wishes to *estimate* the change in the dependent variable (monthly mortgage payment) corresponding to a small change in the independent variable (interest rate per year), occur in many real-life applications. For example:

- An economist would like to know how a small increase in a country's capital expenditure will affect the country's gross domestic output.
- A sociologist would like to know how a small increase in the amount of capital investment in a housing project will affect the crime rate.
- A businesswoman would like to know how raising a product's unit price by a small amount will affect her profit.
- A bacteriologist would like to know how a small increase in the amount of a bactericide will affect a population of bacteria.

To calculate these changes and estimate their effects, we use the *differential* of a function, a concept that will be introduced shortly.

Increments

Let x denote a variable quantity and suppose x changes from x_1 to x_2. This change in x is called the **increment in x** and is denoted by the symbol Δx (read "delta x"). Thus,

$$\Delta x = x_2 - x_1 \qquad \text{(Final value minus initial value)} \tag{9}$$

EXAMPLE 1 Find the increment in x:

a. As x changes from 3 to 3.2 **b.** As x changes from 3 to 2.7

SOLUTION ✓ **a.** Here, $x_1 = 3$ and $x_2 = 3.2$, so

$$\Delta x = x_2 - x_1 = 3.2 - 3 = 0.2$$

b. Here, $x_1 = 3$ and $x_2 = 2.7$. Therefore,

$$\Delta x = x_2 - x_1 = 2.7 - 3 = -0.3$$

■■■■

Observe that Δx plays the same role that h played in Section 10.4.

Now, suppose two quantities, x and y, are related by an equation $y = f(x)$, where f is a function. If x changes from x to $x + \Delta x$, then the

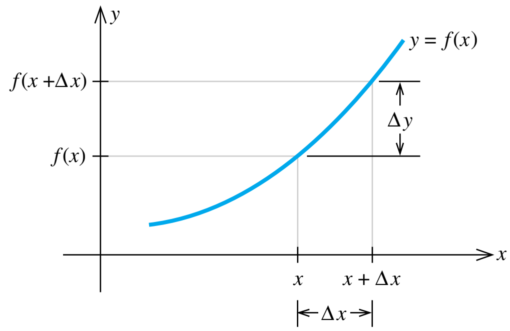


FIGURE 11.18
An increment of Δx in x induces an increment of $\Delta y = f(x + \Delta x) - f(x)$ in y.

corresponding change in y is called the **increment in y**. It is denoted by Δy and is defined in Figure 11.18 by

$$\Delta y = f(x + \Delta x) - f(x) \tag{10}$$

EXAMPLE 2 Let $y = x^3$. Find Δx and Δy:

a. When x changes from 2 to 2.01 **b.** When x changes from 2 to 1.98

SOLUTION ✔ Let $f(x) = x^3$.

a. Here, $\Delta x = 2.01 - 2 = 0.01$. Next,

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) = f(2.01) - f(2)\\ &= (2.01)^3 - 2^3 = 8.120601 - 8 = 0.120601\end{aligned}$$

b. Here, $\Delta x = 1.98 - 2 = -0.02$. Next,

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x) = f(1.98) - f(2)\\ &= (1.98)^3 - 2^3 = 7.762392 - 8 = -0.237608\end{aligned}$$

■■■■

Differentials

We can obtain a relatively quick and simple way of approximating Δy, the change in y due to a small change Δx, by examining the graph of the function f shown in Figure 11.19.

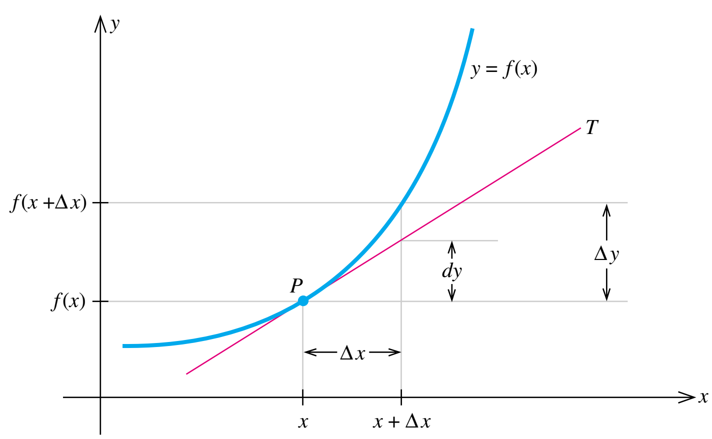


FIGURE 11.19
If Δx is small, dy is a good approximation of Δy.

Observe that near the point of tangency P, the tangent line T is close to the graph of f. Therefore, if Δx is small, then dy is a good approximation of Δy. We can find an expression for dy as follows: Notice that the slope of T is given by

$$\frac{dy}{\Delta x} \qquad \text{(Rise divided by run)}$$

However, the slope of T is given by $f'(x)$. Therefore, we have

$$\frac{dy}{\Delta x} = f'(x)$$

or $dy = f'(x)\Delta x$. Thus, we have the approximation

$$\Delta y \approx dy = f'(x)\Delta x$$

in terms of the derivative of f at x. The quantity dy is called the *differential of y*.

The Differential

Let $y = f(x)$ define a differentiable function of x. Then,

1. The **differential *dx*** of the independent variable x is $dx = \Delta x$.
2. The **differential *dy*** of the dependent variable y is

$$dy = f'(x)\Delta x = f'(x)dx \qquad \textbf{(11)}$$

REMARKS

1. For the independent variable x: There is no difference between Δx and dx—both measure the change in x from x to $x + \Delta x$.
2. For the dependent variable y: Δy measures the *actual* change in y as x changes from x to $x + \Delta x$, whereas dy measures the *approximate* change in y corresponding to the same change in x.
3. The differential dy depends on both x and dx, but for fixed x, dy is a linear function of dx.

EXAMPLE 3 Let $y = x^3$.

a. Find the differential dy of y.
b. Use dy to approximate Δy when x changes from 2 to 2.01.
c. Use dy to approximate Δy when x changes from 2 to 1.98.
d. Compare the results of part (b) with those of Example 2.

SOLUTION ✓ a. Let $f(x) = x^3$. Then,

$$dy = f'(x)\,dx = 3x^2dx$$

b. Here, $x = 2$ and $dx = 2.01 - 2 = 0.01$. Therefore,

$$dy = 3x^2\,dx = 3(2)^2(0.01) = 0.12$$

c. Here, $x = 2$ and $dx = 1.98 - 2 = -0.02$. Therefore,

$$dy = 3x^2\,dx = 3(2)^2(-0.02) = -0.24$$

d. As you can see, both approximations 0.12 and −0.24 are quite close to the actual changes of Δy obtained in Example 2: 0.120601 and −0.237608.

Observe how much easier it is to find an approximation to the exact change in a function with the help of the differential, rather than calculating the exact change in the function itself. In the following examples, we take advantage of this fact.

EXAMPLE 4 Approximate the value of $\sqrt{26.5}$ using differentials. Verify your result using the $\boxed{\sqrt{\ }}$ key on your calculator.

SOLUTION ✔ Since we want to compute the square root of a number, let's consider the function $y = f(x) = \sqrt{x}$. Since 25 is the number nearest 26.5 whose square root is readily recognized, let's take $x = 25$. We want to know the change in y, Δy, as x changes from $x = 25$ to $x = 26.5$, an increase of $\Delta x = 1.5$ units. Using Equation (11), we find

$$\begin{aligned}\Delta y \approx dy &= f'(x)\Delta x\\ &= \left[\left.\frac{1}{2\sqrt{x}}\right|_{x=25}\right]\cdot(1.5) = \left(\frac{1}{10}\right)(1.5) = 0.15\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{26.5} - \sqrt{25} &= \Delta y \approx 0.15\\ \sqrt{26.5} &\approx \sqrt{25} + 0.15 = 5.15\end{aligned}$$

The exact value of $\sqrt{26.5}$, rounded off to five decimal places, is 5.14782. Thus, the error incurred in the approximation is 0.00218.

APPLICATIONS

EXAMPLE 5 The total cost incurred in operating a certain type of truck on a 500-mile trip, traveling at an average speed of v mph, is estimated to be

$$C(v) = 125 + v + \frac{4500}{v}$$

dollars. Find the approximate change in the total operating cost when the average speed is increased from 55 mph to 58 mph.

SOLUTION ✓ With $v = 55$ and $\Delta v = dv = 3$, we find

$$\Delta C \approx dC = C'(v)dv = \left(1 - \frac{4500}{v^2}\right)\bigg|_{v=55} \cdot 3$$
$$= \left(1 - \frac{4500}{3025}\right)(3) \approx -1.46$$

so the total operating cost is found to decrease by \$1.46. This might explain why so many independent truckers often exceed the 55 mph speed limit.

EXAMPLE 6 The relationship between the amount of money x spent by Cannon Precision Instruments on advertising and Cannon's total sales $S(x)$ is given by the function

$$S(x) = -0.002x^3 + 0.6x^2 + x + 500 \qquad (0 \leq x \leq 200)$$

where x is measured in thousands of dollars. Use differentials to estimate the change in Cannon's total sales if advertising expenditures are increased from \$100,000 ($x = 100$) to \$105,000 ($x = 105$).

SOLUTION ✓ The required change in sales is given by

$$\Delta S \approx dS = S'(100)dx$$
$$= -0.006x^2 + 1.2x + 1\big|_{x=100} \cdot (5) \qquad (dx = 105 - 100 = 5)$$
$$= (-60 + 120 + 1)(5) = 305$$

—that is, an increase of \$305,000.

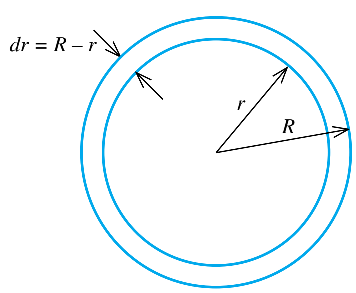
EXAMPLE 7

The Rings of Neptune

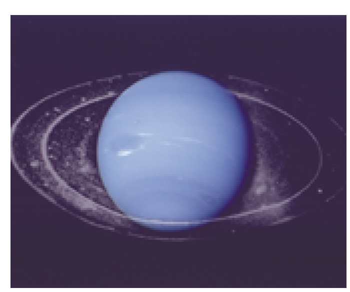
a. A ring has an inner radius of r units and an outer radius of R units, where $(R - r)$ is small in comparison to r (Figure 11.20a). Use differentials to estimate the area of the ring.

b. Recent observations, including those of *Voyager I* and *II,* showed that Neptune's ring system is considerably more complex than had been believed. For one thing, it is made up of a large number of distinguishable rings rather than one continuous great ring as previously thought (Figure 11.20b). The

FIGURE 11.20



(a) The area of the ring is the circumference of the inner circle times the thickness.



(b) Neptune and its rings.

outermost ring, 1989N1R, has an inner radius of approximately 62,900 kilometers (measured from the center of the planet), and a radial width of approximately 50 kilometers. Using these data, estimate the area of the ring.

SOLUTION ✔

a. Using the fact that the area of a circle of radius x is $A = f(x) = \pi x^2$, we find

$$\begin{aligned}\pi R^2 - \pi r^2 &= f(R) - f(r)\\ &= \Delta A \qquad \text{(Remember, } \Delta A = \text{change in } f \text{ when } x \text{ changes from } x = r \text{ to } x = R.)\\ &\approx dA\\ &= f'(r)dr\end{aligned}$$

where $dr = R - r$. So, we see that the area of the ring is approximately $2\pi r(R - r)$ square units. In words, the area of the ring is approximately equal to

Circumference of the inner circle × Thickness of the ring

b. Applying the results of part (a) with $r = 62{,}900$ and $dr = 50$, we find that the area of the ring is approximately $2\pi(62{,}900)(50)$, or 19,760,618 square kilometers, which is roughly 4% of Earth's surface. ■■■■

Before looking at the next example, we need to familiarize ourselves with some terminology. If a quantity with exact value q is measured or calculated with an error of Δq, then the quantity $\Delta q/q$ is called the relative error in the measurement or calculation of q. If the quantity $\Delta q/q$ is expressed as a percentage, it is then called the percentage error. Because Δq is approximated by dq, we normally approximate the relative error $\Delta q/q$ by dq/q.

EXAMPLE 8

Suppose the radius of a ball-bearing is measured to be 0.5 inch, with a maximum error of ± 0.0002 inch. Then, the relative error in r is

$$\frac{dr}{r} = \frac{\pm 0.0002}{0.5} = \pm 0.0004$$

and the percentage error is $\pm 0.04\%$. ■■■■

EXAMPLE 9

Suppose the side of a cube is measured with a maximum percentage error of 2%. Use differentials to estimate the maximum percentage error in the calculated volume of the cube.

SOLUTION ✔

Suppose the side of the cube is x, so its volume is

$$V = x^3$$

We are given that $\left|\frac{dx}{x}\right| \le 0.02$. Now,

$$dV = 3x^2 dx$$

and so

$$\frac{dV}{V} = \frac{3x^2 dx}{x^3} = 3\frac{dx}{x}$$

Therefore,

$$\left|\frac{dV}{V}\right| = 3\left|\frac{dx}{x}\right| \leq 3(0.02) = 0.06$$

and we see that the maximum percentage error in the measurement of the volume of the cube is 6%. ■■■■

Finally, we want to point out that if at some point in reading this section you have a sense of déjà vu, do not be surprised, because the notion of the differential was first used in Section 11.4 (see Example 1). There we took $\Delta x = 1$ since we were interested in finding the marginal cost when the level of production was increased from $x = 250$ to $x = 251$. If we had used differentials, we would have found

$$C(251) - C(250) \approx C'(250)dx$$

so that taking $dx = \Delta x = 1$, we have $C(251) - C(250) \approx C'(250)$, which agrees with the result obtained in Example 1. Thus, in Section 11.4, we touched upon the notion of the differential, albeit in the special case in which $dx = 1$.

Self-Check Exercises 11.7

1. Find the differential of $f(x) = \sqrt{x} + 1$.
2. A certain country's government economists have determined that the demand equation for corn in that country is given by

$$p = f(x) = \frac{125}{x^2 + 1}$$

where p is expressed in dollars per bushel and x, the quantity demanded per year, is measured in billions of bushels. The economists are forecasting a harvest of 6 billion bushels for the year. If the actual production of corn were 6.2 billion bushels for the year instead, what would be the approximate drop in the predicted price of corn per bushel?

Solutions to Self-Check Exercises 11.7 can be found on page 782.

11.7 Exercises

In Exercises 1–14, find the differential of the given function.

1. $f(x) = 2x^2$
2. $f(x) = 3x^2 + 1$
3. $f(x) = x^3 - x$
4. $f(x) = 2x^3 + x$
5. $f(x) = \sqrt{x + 1}$
6. $f(x) = \dfrac{3}{\sqrt{x}}$
7. $f(x) = 2x^{3/2} + x^{1/2}$
8. $f(x) = 3x^{5/6} + 7x^{2/3}$
9. $f(x) = x + \dfrac{2}{x}$
10. $f(x) = \dfrac{3}{x - 1}$
11. $f(x) = \dfrac{x - 1}{x^2 + 1}$
12. $f(x) = \dfrac{2x^2 + 1}{x + 1}$
13. $f(x) = \sqrt{3x^2 - x}$
14. $f(x) = (2x^2 + 3)^{1/3}$
15. Let f be a function defined by

$$y = f(x) = x^2 - 1$$

a. Find the differential of f.
b. Use your result from part (a) to find the approximate change in y if x changes from 1 to 1.02.
c. Find the actual change in y if x changes from 1 to 1.02 and compare your result with that obtained in part (b).

16. Let f be a function defined by

$$y = f(x) = 3x^2 - 2x + 6$$

a. Find the differential of f.
b. Use your result from part (a) to find the approximate change in y if x changes from 2 to 1.97.
c. Find the actual change in y if x changes from 2 to 1.97 and compare your result with that obtained in part (b).

17. Let f be a function defined by

$$y = f(x) = \frac{1}{x}$$

a. Find the differential of f.
b. Use your result from part (a) to find the approximate change in y if x changes from -1 to -0.95.
c. Find the actual change in y if x changes from -1 to -0.95 and compare your result with that obtained in part (b).

18. Let f be a function defined by

$$y = f(x) = \sqrt{2x + 1}$$

a. Find the differential of f.
b. Use your result from part (a) to find the approximate change in y if x changes from 4 to 4.1.
c. Find the actual change in y if x changes from 4 to 4.1 and compare your result with that obtained in part (b).

In Exercises 19–26, use differentials to approximate the given quantity.

19. $\sqrt{10}$ **20.** $\sqrt{17}$ **21.** $\sqrt{49.5}$

22. $\sqrt{99.7}$ **23.** $\sqrt[3]{7.8}$ **24.** $\sqrt[4]{81.6}$

25. $\sqrt{0.089}$ **26.** $\sqrt[3]{0.00096}$

27. Use a differential to approximate $\sqrt{4.02} + \dfrac{1}{\sqrt{4.02}}$.

Hint: Let $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ and compute dy with $x = 4$ and $dx = 0.02$.

28. Use a differential to approximate $\dfrac{2(4.98)}{(4.98)^2 + 1}$.

Hint: Study the hint for Exercise 27.

A calculator is recommended for the remainder of this exercise set.

29. Error Estimation The length of each edge of a cube is 12 cm, with a possible error in measurement of 0.02 cm. Use differentials to estimate the error that might occur when the volume of the cube is calculated.

30. Estimating the Amount of Paint Required A coat of paint of thickness 0.05 cm is to be applied uniformly to the faces of a cube of edge 30 cm. Use differentials to find the approximate amount of paint required for the job.

31. Error Estimation A hemisphere-shaped dome of radius 60 ft is to be coated with a layer of rust-proofer before painting. Use differentials to estimate the amount of rust-proofer needed if the coat is to be 0.01 in. thick.
Hint: The volume of a hemisphere of radius r is $V = \frac{2}{3}\pi r^3$.

32. Growth of a Cancerous Tumor The volume of a spherical cancer tumor is given by

$$V(r) = \frac{4}{3}\pi r^3$$

If the radius of a tumor is estimated at 1.1 cm, with a maximum error in measurement of 0.005 cm, determine the error that might occur when the volume of the tumor is calculated.

33. Unclogging Arteries Research done in the 1930s by the French physiologist Jean Poiseuille showed that the resistance R of a blood vessel of length l and radius r is $R = kl/r^4$, where k is a constant. Suppose a dose of the drug TPA increases r by 10%. How will this affect the resistance R? Assume that l is constant.

34. Gross Domestic Product An economist has determined that a certain country's gross domestic product (GDP) is approximated by the function $f(x) = 640x^{1/5}$, where $f(x)$ is measured in billions of dollars and x is the capital outlay in billions of dollars. Use differentials to estimate the change in the country's GDP if the country's capital expenditure changes from \$243 billion to \$248 billion.

35. Learning Curves The length of time (in seconds) a certain individual takes to learn a list of n items is approximated by

$$f(n) = 4n\sqrt{n - 4}$$

Use differentials to approximate the additional time it takes the individual to learn the items on a list when n is increased from 85 to 90 items.

36. **EFFECT OF ADVERTISING ON PROFITS** The relationship between Cunningham Realty's quarterly profits, $P(x)$, and the amount of money x spent on advertising per quarter is described by the function

$$P(x) = -\frac{1}{8}x^2 + 7x + 30 \qquad (0 \le x \le 50)$$

where both $P(x)$ and x are measured in thousands of dollars. Use differentials to estimate the increase in profits when advertising expenditure each quarter is increased from \$24,000 to \$26,000.

37. **EFFECT OF MORTGAGE RATES ON HOUSING STARTS** A study prepared for the National Association of Realtors estimates that the number of housing starts per year over the next 5 yr will be

$$N(r) = \frac{7}{1 + 0.02r^2}$$

million units, where r (percent) is the mortgage rate. Use differentials to estimate the decrease in the number of housing starts when the mortgage rate is increased from 12% to 12.5%.

38. **SUPPLY-PRICE** The supply equation for a certain brand of transistor radio is given by

$$p = s(x) = 0.3\sqrt{x} + 10$$

where x is the quantity supplied and p is the unit price in dollars. Use differentials to approximate the change in price when the quantity supplied is increased from 10,000 units to 10,500 units.

39. **DEMAND-PRICE** The demand function for the Sentinel smoke alarm is given by

$$p = d(x) = \frac{30}{0.02x^2 + 1}$$

where x is the quantity demanded (in units of a thousand) and p is the unit price in dollars. Use differentials to estimate the change in the price p when the quantity demanded changes from 5000 to 5500 units/wk.

40. **SURFACE AREA OF AN ANIMAL** Animal physiologists use the formula

$$S = kW^{2/3}$$

to calculate an animal's surface area (in square meters) from its weight W (in kilograms), where k is a constant that depends on the animal under consideration. Suppose a physiologist calculates the surface area of a horse ($k = 0.1$). If the horse's weight is estimated at 300 kg, with a maximum error in measurement of 0.6 kg, determine the percentage error in the calculation of the horse's surface area.

41. **FORECASTING PROFITS** The management of Trappee and Sons, Inc., forecast that they will sell 200,000 cases of their Texa-Pep hot sauce next year. Their annual profit is described by

$$P(x) = -0.000032x^3 + 6x - 100$$

thousand dollars, where x is measured in thousands of cases. If the maximum error in the forecast is 15%, determine the corresponding error in Trappee's profits.

42. **FORECASTING COMMODITY PRICES** A certain country's government economists have determined that the demand equation for soybeans in that country is given by

$$p = f(x) = \frac{55}{2x^2 + 1}$$

where p is expressed in dollars per bushel and x, the quantity demanded per year, is measured in billions of bushels. The economists are forecasting a harvest of 1.8 billion bushels for the year, with a maximum error of 15% in their forecast. Determine the corresponding maximum error in the predicted price per bushel of soybeans.

43. **CRIME STUDIES** A sociologist has found that the number of serious crimes in a certain city per year is described by the function

$$N(x) = \frac{500(400 + 20x)^{1/2}}{(5 + 0.2x)^2}$$

where x (in cents per dollar deposited) is the level of reinvestment in the area in conventional mortgages by the city's ten largest banks. Use differentials to estimate the change in the number of crimes if the level of reinvestment changes from 20 cents/dollar deposited to 22 cents/dollar deposited.

44. **FINANCING A HOME** The Millers are planning to buy a home in the near future and estimate that they will need a 30-yr fixed-rate mortgage for \$120,000. Their monthly payment P (in dollars) can be computed using

(continued on page 782)

Using Technology

Finding the Differential of a Function

The calculation of the differential of f at a given value of x involves the evaluation of the derivative of f at that point and can be facilitated through the use of the numerical derivative function.

EXAMPLE 1 Use dy to approximate Δy if $y = x^2(2x^2 + x + 1)^{2/3}$ and x changes from 2 to 1.98.

SOLUTION ✔ Let $f(x) = x^2(2x^2 + x + 1)^{2/3}$. Since $dx = 1.98 - 2 = -0.02$, we find the required approximation to be

$$dy = f'(2) \cdot (-0.02)$$

But using the numerical derivative operation, we find

$$f'(2) = 30.5758132855$$

and so

$$dy = (-0.02)(30.5758132855) = -0.611516266$$

EXAMPLE 2 The Meyers are considering the purchase of a house in the near future and estimate that they will need a loan of \$120,000. Based on a 30-year conventional mortgage with an interest rate of r per year, their monthly repayment will be

$$P = \frac{10{,}000r}{1 - \left(1 + \dfrac{r}{12}\right)^{-360}}$$

dollars. If the interest rate increases from the present rate of 10%/year to 10.2% per year between now and the time the Meyers decide to secure the loan, approximately how much more per month will their mortgage payment be?

SOLUTION ✔ Let's write

$$P = f(r) = \frac{10{,}000r}{1 - \left(1 + \dfrac{r}{12}\right)^{-360}}$$

Then the increase in the mortgage payment will be approximately

$$\begin{aligned} dP &= f'(0.1)dr = f'(0.1)(0.002) && \text{(Since } dr = 0.102 - 0.1\text{)} \\ &= (8867.59947979)(0.002) \approx 17.7352 && \text{(Using the numerical derivative operation)} \end{aligned}$$

or approximately \$17.74 per month.

Exercises

In Exercises 1–6, use dy to approximate Δy for the function $y = f(x)$ when x changes from $x = a$ to $x = b$.

1. $f(x) = 0.21x^7 - 3.22x^4 + 5.43x^2 + 1.42x + 12.42$; $a = 3$, $b = 3.01$

2. $f(x) = \dfrac{0.2x^2 + 3.1}{1.2x + 1.3}$; $a = 2$, $b = 1.96$

3. $f(x) = \sqrt{2.2x^2 + 1.3x + 4}$; $a = 1$, $b = 1.03$

4. $f(x) = x\sqrt{2x^3 - x + 4}$; $a = 2$, $b = 1.98$

5. $f(x) = \dfrac{\sqrt{x^2 + 4}}{x - 1}$; $a = 4$, $b = 4.1$

6. $f(x) = 2.1x^2 + \dfrac{3}{\sqrt{x}} + 5$; $a = 3$, $b = 2.95$

7. **Calculating Mortgage Payments** Refer to Example 2. How much more per month will the Meyers' mortgage payment be if the interest rate increases from 10% to 10.3%/year? To 10.4%/year? To 10.5%/year?

8. **Estimating the Area of a Ring of Neptune** The ring 1989N2R of the planet Neptune has an inner radius of approximately 53,200 km (measured from the center of the planet) and a radial width of 15 km. Use differentials to estimate the area of the ring.

9. **Effect of Price Increase on Quantity Demanded** The quantity demanded per week of the Alpha Sports Watch, x (in thousands), is related to its unit price of p dollars by the equation

$$x = f(p) = 10\sqrt{\frac{50 - p}{p}} \qquad (0 \le p \le 50)$$

Use differentials to find the decrease in the quantity of the watches demanded per week if the unit price is increased from \$40 to \$42.

the formula

$$P = \frac{10{,}000r}{1 - \left(1 + \frac{r}{12}\right)^{-360}}$$

where r is the interest rate per year.

a. Find the differential of P.

b. If the interest rate increases from the present rate of 9%/year to 9.2%/year between now and the time the Millers decide to secure the loan, approximately how much more will their monthly mortgage payment be? How much more will it be if the interest rate increases to 9.3%/year? To 9.4%/year? To 9.5%/year?

In Exercises 45 and 46, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

45. If $y = ax + b$ where a and b are constants, then $\Delta y = dy$.

46. If $A = f(x)$, then the percentage change in A is

$$\frac{100f'(x)}{f(x)}\,dx$$

Solutions to Self-Check Exercises 11.7

1. We find

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Therefore, the required differential of f is

$$dy = \frac{1}{2\sqrt{x}}\,dx$$

2. We first compute the differential

$$dp = -\frac{250x}{(x^2+1)^2}\,dx$$

Next, using Equation (11) with $x = 6$ and $dx = 0.2$, we find

$$\Delta p \approx dp = -\frac{250(6)}{(36+1)^2}(0.2) = -0.22$$

or a drop in price of 22 cents/bushel.

CHAPTER 11 Summary of Principal Formulas and Terms

Formulas

1. Derivative of a constant	$\dfrac{d}{dx}(c) = 0$, c a constant
2. Power rule	$\dfrac{d}{dx}(x^n) = nx^{n-1}$
3. Constant multiple rule	$\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$, c a constant
4. Sum rule	$\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$
5. Product rule	$\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
6. Quotient rule	$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$
7. Chain rule	$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
8. General power rule	$\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$
9. Average cost function	$\overline{C}(x) = \dfrac{C(x)}{x}$
10. Revenue function	$R(x) = px$
11. Profit function	$P(x) = R(x) - C(x)$
12. Elasticity of demand	$E(p) = -\dfrac{pf'(p)}{f(p)}$
13. Differential of y	$dy = f'(x)dx$

Terms

marginal cost function
marginal average cost function
marginal revenue function
marginal profit function
elastic demand
unitary demand
inelastic demand
second derivative of f
implicit differentiation
related rates

CHAPTER 11 REVIEW EXERCISES

In Exercises 1–30, find the derivative of the given function.

1. $f(x) = 3x^5 - 2x^4 + 3x^2 - 2x + 1$
2. $f(x) = 4x^6 + 2x^4 + 3x^2 - 2$
3. $g(x) = -2x^{-3} + 3x^{-1} + 2$
4. $f(t) = 2t^2 - 3t^3 - t^{-1/2}$
5. $g(t) = 2t^{-1/2} + 4t^{-3/2} + 2$
6. $h(x) = x^2 + \dfrac{2}{x}$
7. $f(t) = t + \dfrac{2}{t} + \dfrac{3}{t^2}$
8. $g(s) = 2s^2 - \dfrac{4}{s} + \dfrac{2}{\sqrt{s}}$
9. $h(x) = x^2 - \dfrac{2}{x^{3/2}}$
10. $f(x) = \dfrac{x+1}{2x-1}$
11. $g(t) = \dfrac{t^2}{2t^2+1}$
12. $h(t) = \dfrac{\sqrt{t}}{\sqrt{t}+1}$
13. $f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$
14. $f(t) = \dfrac{t}{2t^2+1}$
15. $f(x) = \dfrac{x^2(x^2+1)}{x^2-1}$
16. $f(x) = (2x^2 + x)^3$
17. $f(x) = (3x^3 - 2)^8$
18. $h(x) = (\sqrt{x} + 2)^5$
19. $f(t) = \sqrt{2t^2+1}$
20. $g(t) = \sqrt[3]{1-2t^3}$
21. $s(t) = (3t^2 - 2t + 5)^{-2}$
22. $f(x) = (2x^3 - 3x^2 + 1)^{-3/2}$
23. $h(x) = \left(x + \dfrac{1}{x}\right)^2$
24. $h(x) = \dfrac{1+x}{(2x^2+1)^2}$
25. $h(t) = (t^2 + t)^4(2t^2)$
26. $f(x) = (2x + 1)^3(x^2 + x)^2$
27. $g(x) = \sqrt{x}(x^2-1)^3$
28. $f(x) = \dfrac{x}{\sqrt{x^3+2}}$
29. $h(x) = \dfrac{\sqrt{3x+2}}{4x-3}$
30. $f(t) = \dfrac{\sqrt{2t+1}}{(t+1)^3}$

In Exercises 31–36, find the second derivative of the given function.

31. $f(x) = 2x^4 - 3x^3 + 2x^2 + x + 4$
32. $g(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$
33. $h(t) = \dfrac{t}{t^2+4}$
34. $f(x) = (x^3 + x + 1)^2$
35. $f(x) = \sqrt{2x^2+1}$
36. $f(t) = t(t^2 + 1)^3$

In Exercises 37–42, find *dy/dx* by implicit differentiation.

37. $6x^2 - 3y^2 = 9$
38. $2x^3 - 3xy = 4$
39. $y^3 + 3x^2 = 3y$
40. $x^2 + 2x^2y^2 + y^2 = 10$
41. $x^2 - 4xy - y^2 = 12$
42. $3x^2y - 4xy + x - 2y = 6$
43. Let $f(x) = 2x^3 - 3x^2 - 16x + 3$.
    a. Find the points on the graph of f at which the slope of the tangent line is equal to -4.
    b. Find the equation(s) of the tangent line(s) of part (a).
44. Let $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 4x + 1$.
    a. Find the points on the graph of f at which the slope of the tangent line is equal to -2.
    b. Find the equation(s) of the tangent line(s) of part (a).
45. Find an equation of the tangent line to the graph of $y = \sqrt{4-x^2}$ at the point $(1, \sqrt{3})$.
46. Find an equation of the tangent line to the graph of $y = x(x+1)^5$ at the point (1, 32).
47. Find the third derivative of the function

$$f(x) = \frac{1}{2x-1}$$

    What is its domain?
48. **WORLDWIDE NETWORKED PCS** The number of worldwide networked PCs (in millions) is given by

$$N(t) = 3.136x^2 + 3.954x + 116.468 \qquad (0 \le t \le 9)$$

    where t is measured in years, with $t = 0$ corresponding to the beginning of 1991.
    a. How many worldwide networked PCs were there at the beginning of 1997?
    b. How fast was the number of worldwide networked PCs changing at the beginning of 1997?
49. The number of subscribers to CNC Cable Television in the town of Randolph is approximated by the function

$$N(x) = 1000(1 + 2x)^{1/2} \qquad (1 \le x \le 30)$$

    where $N(x)$ denotes the number of subscribers to the service in the xth week. Find the rate of increase in the number of subscribers at the end of the 12th week.
50. The total weekly cost in dollars incurred by the Herald Record Company in pressing x video discs is given by the total cost function

$$C(x) = 2500 + 2.2x \qquad (0 \le x \le 8000)$$

a. What is the marginal cost when $x = 1000$? When $x = 2000$?
b. Find the average cost function $\overline{C}$ and the marginal average cost function $\overline{C}'$.
c. Using the results from part (b), show that the average cost incurred by Herald in pressing a video disc approaches \$2.20/disc when the level of production is high enough.

51. The marketing department of Telecon Corporation has determined that the demand for their cordless phones obeys the relationship

$$p = -0.02x + 600 \qquad (0 \le x \le 30{,}000)$$

where p denotes the phone's unit price (in dollars) and x denotes the quantity demanded.
a. Find the revenue function R.
b. Find the marginal revenue function R'.
c. Compute $R'(10{,}000)$ and interpret your result.

52. The weekly demand for the Lectro-Copy photocopying machine is given by the demand equation

$$p = 2000 - 0.04x \qquad (0 \le x \le 50{,}000)$$

where p denotes the wholesale unit price in dollars and x denotes the quantity demanded. The weekly total cost function for manufacturing these copiers is given by

$$C(x) = 0.000002x^3 - 0.02x^2 + 1000x + 120{,}000$$

where $C(x)$ denotes the total cost incurred in producing x units.
a. Find the revenue function R, the profit function P, and the average cost function $\overline{C}$.
b. Find the marginal cost function C', the marginal revenue function R', the marginal profit function P', and the marginal average cost function $\overline{C}'$.
c. Compute $C'(3000)$, $R'(3000)$, and $P'(3000)$.
d. Compute $\overline{C}'(5000)$ and $\overline{C}'(8000)$ and interpret your results.