13

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

The exponential function is, without doubt, the most important function in mathematics and its applications. After a brief introduction to the exponential function and its *inverse,* the logarithmic function, we learn how to differentiate such functions. This lays the foundation for exploring the many applications involving exponential functions. For example, we look at the role played by exponential functions in computing earned interest on a bank account, in studying the growth of a bacteria population in the laboratory, in studying the way radioactive matter decays, in studying the rate at which a factory worker learns a certain process, and in studying the rate at which a communicable disease is spread over time.

How many bacteria will there be in a culture at the end of a certain period of time? How fast will the bacteria population be growing at the end of that time? Example 1, page 928, answers these questions.

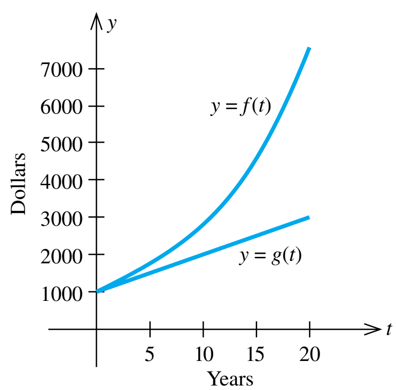
13.1 Exponential Functions

EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

Suppose you deposit a sum of \$1000 in an account earning interest at the rate of 10% per year *compounded continuously* (the way most financial institutions compute interest). Then, the accumulated amount at the end of t years $(0 \leq t \leq 20)$ is described by the function f, whose graph appears in Figure 13.1.* Such a function is called an *exponential function.* Observe that the graph of f rises rather slowly at first but very rapidly as time goes by. For purposes of comparison, we have also shown the graph of the function $y = g(t) = 1000(1 + 0.10t)$, giving the accumulated amount for the same principal (\$1000) but earning *simple* interest at the rate of 10% per year. The moral of the story: It is never too early to save.

Exponential functions play an important role in many real-world applications, as you will see throughout this chapter.

FIGURE 13.1
Under continuous compounding, a sum of money grows exponentially.



Observe that whenever b is a positive number and n is any real number, the expression b^n is a real number. This enables us to define an exponential function as follows:

Exponential Function

The function defined by

$$f(x) = b^x \qquad (b > 0, b \neq 1)$$

is called an **exponential function with base b and exponent x.** The domain of f is the set of all real numbers.

* We will derive the rule for f later in this section.

For example, the exponential function with base 2 is the function

$$f(x) = 2^x$$

with domain $(-\infty, \infty)$. The values of $f(x)$ for selected values of x follow:

$$f(3) = 2^3 = 8, \qquad f\left(\frac{3}{2}\right) = 2^{3/2} = 2 \cdot 2^{1/2} = 2\sqrt{2}, \qquad f(0) = 2^0 = 1,$$

$$f(-1) = 2^{-1} = \frac{1}{2}, \qquad f\left(-\frac{2}{3}\right) = 2^{-2/3} = \frac{1}{2^{2/3}} = \frac{1}{\sqrt[3]{4}}$$

Computations involving exponentials are facilitated by the laws of exponents. These laws were stated in Section 9.1, and you might want to review the material there. For convenience, however, we will restate these laws.

Laws of Exponents

Let a and b be positive numbers and let x and y be real numbers. Then,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

The use of the laws of exponents is illustrated in the next example.

EXAMPLE 1

a. $16^{7/4} \cdot 16^{-1/2} = 16^{7/4-1/2} = 16^{5/4} = 2^5 = 32$ (Law 1)

b. $\dfrac{8^{5/3}}{8^{-1/3}} = 8^{5/3-(-1/3)} = 8^2 = 64$ (Law 2)

c. $(64^{4/3})^{-1/2} = 64^{(4/3)(-1/2)} = 64^{-2/3}$

$$= \frac{1}{64^{2/3}} = \frac{1}{(64^{1/3})^2} = \frac{1}{4^2} = \frac{1}{16} \qquad \text{(Law 3)}$$

d. $(16 \cdot 81)^{-1/4} = 16^{-1/4} \cdot 81^{-1/4} = \dfrac{1}{16^{1/4}} \cdot \dfrac{1}{81^{1/4}} = \dfrac{1}{2} \cdot \dfrac{1}{3} = \dfrac{1}{6}$ (Law 4)

e. $\left(\dfrac{3^{1/2}}{2^{1/3}}\right)^4 = \dfrac{3^{4/2}}{2^{4/3}} = \dfrac{9}{2^{4/3}}$ (Law 5)

■■■■

EXAMPLE 2 Let $f(x) = 2^{2x-1}$. Find the value of x for which $f(x) = 16$.

SOLUTION We want to solve the equation

$$2^{2x-1} = 16 = 2^4$$

But this equation holds if and only if

$$2x - 1 = 4 \qquad (b^m = b^n \Rightarrow m = n)$$

giving $x = \frac{5}{2}$.

■■■■

Exponential functions play an important role in mathematical analysis. Because of their special characteristics, they are some of the most useful functions and are found in virtually every field where mathematics is applied. To mention a few examples: Under ideal conditions the number of bacteria present at any time t in a culture may be described by an exponential function of t; radioactive substances decay over time in accordance with an "exponential" law of decay; money left on fixed deposit and earning compound interest grows exponentially; and some of the most important distribution functions encountered in statistics are exponential.

Let's begin our investigation into the properties of exponential functions by studying their graphs.

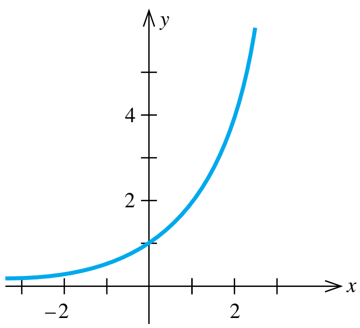
EXAMPLE 3 Sketch the graph of the exponential function $y = 2^x$.

SOLUTION ✓ First, as discussed earlier, the domain of the exponential function $y = f(x) = 2^x$ is the set of real numbers. Next, putting $x = 0$ gives $y = 2^0 = 1$, the y-intercept of f. There is no x-intercept since there is no value of x for which $y = 0$. To find the range of f, consider the following table of values:

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	1/32	1/16	1/8	1/4	1/2	1	2	4	8	16	32

We see from these computations that 2^x decreases and approaches zero as x decreases without bound and that 2^x increases without bound as x increases without bound. Thus, the range of f is the interval $(0, \infty)$—that is, the set of positive real numbers. Finally, we sketch the graph of $y = f(x) = 2^x$ in Figure 13.2. ■■■■

FIGURE 13.2
The graph of $y = 2^x$



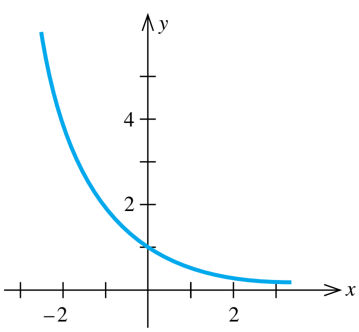
EXAMPLE 4 Sketch the graph of the exponential function $y = (1/2)^x$.

SOLUTION ✓ The domain of the exponential function $y = (1/2)^x$ is the set of all real numbers. The y-intercept is $(1/2)^0 = 1$; there is no x-intercept since there is no value of x for which $y = 0$. From the following table of values

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	32	16	8	4	2	1	1/2	1/4	1/8	1/16	1/32

we deduce that $(1/2)^x = 1/2^x$ increases without bound as x decreases without bound and that $(1/2)^x$ decreases and approaches zero as x increases without bound. Thus, the range of f is the interval $(0, \infty)$. The graph of $y = f(x) = (1/2)^x$ is sketched in Figure 13.3. ■■■■

FIGURE 13.3
The graph of $y = (1/2)^x$



The functions $y = 2^x$ and $y = (1/2)^x$, whose graphs you studied in Examples 3 and 4, are special cases of the exponential function $y = f(x) = b^x$, obtained by setting $b = 2$ and $b = 1/2$, respectively. In general, the exponential function $y = b^x$ with $b > 1$ has a graph similar to $y = 2^x$, whereas the graph of $y = b^x$ for $0 < b < 1$ is similar to that of $y = (1/2)^x$ (Exercises 27 and 28). When $b = 1$, the function $y = b^x$ reduces to the constant function $y = 1$. For comparison, the graphs of all three functions are sketched in Figure 13.4.

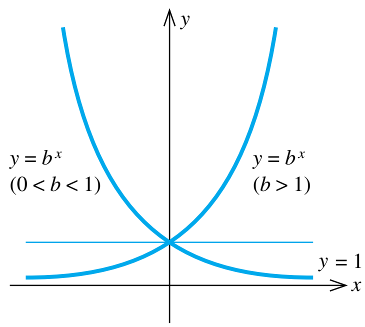


FIGURE 13.4
$y = b^x$ is an increasing function of x if $b > 1$, a constant function if $b = 1$, and a decreasing function if $0 < b < 1$.

Properties of the Exponential Function

The exponential function $y = b^x$ $(b > 0, b \neq 1)$ has the following properties:

1. Its domain is $(-\infty, \infty)$.
2. Its range is $(0, \infty)$.
3. Its graph passes through the point $(0, 1)$.
4. It is continuous on $(-\infty, \infty)$.
5. It is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.

The Base e

Exponential functions to the base e, where e is an irrational number whose value is 2.7182818 . . . , play an important role in both theoretical and applied problems. It can be shown, although we will not do so here, that

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m \tag{1}$$

However, you may convince yourself of the plausibility of this definition of the number e by examining Table 13.1, which may be constructed with the help of a calculator.

Table 13.1

m	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \frac{1}{m}\right)^m$	2.59374	2.70481	2.71692	2.71815	2.71827	2.71828

Exploring with Technology

To obtain a visual confirmation of the fact that the expression $(1 + 1/m)^m$ approaches the number $e = 2.71828\ldots$ as m increases without bound, plot the graph of $f(x) = (1 + 1/x)^x$ in a suitable viewing rectangle and observe that $f(x)$ approaches $2.71828\ldots$ as x increases without bound. Use **ZOOM** and **TRACE** to find the value of $f(x)$ for large values of x.

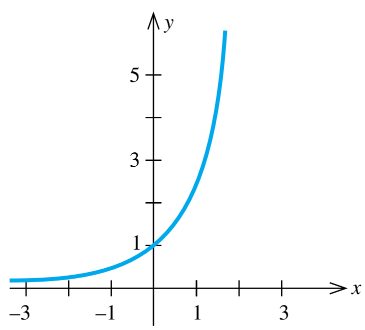
EXAMPLE 5 Sketch the graph of the function $y = e^x$.

SOLUTION Since $e > 1$, it follows from our previous discussion that the graph of $y = e^x$ is similar to the graph of $y = 2^x$ (see Figure 13.2). With the aid of a calculator, we obtain the following table:

x	-3	-2	-1	0	1	2	3
y	0.05	0.14	0.37	1	2.72	7.39	20.09

The graph of $y = e^x$ is sketched in Figure 13.5.

FIGURE 13.5
The graph of $y = e^x$



Next, we consider another exponential function to the base e that is closely related to the previous function and is particularly useful in constructing models that describe "exponential decay."

EXAMPLE 6 Sketch the graph of the function $y = e^{-x}$.

SOLUTION Since $e > 1$, it follows that $0 < 1/e < 1$, so $f(x) = e^{-x} = 1/e^x = (1/e)^x$ is an exponential function with base less than 1. Therefore, it has a graph similar to that of the exponential function $y = (1/2)^x$. As before, we construct the following table of values of $y = e^{-x}$ for selected values of x:

x	-3	-2	-1	0	1	2	3
y	20.09	7.39	2.72	1	0.37	0.14	0.05

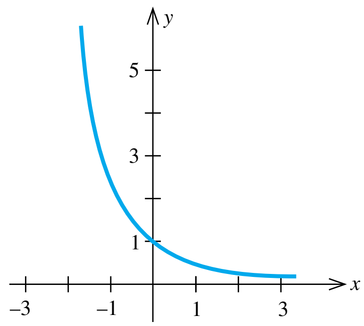


FIGURE 13.6
The graph of $y = e^{-x}$

Using this table, we sketch the graph of $y = e^{-x}$ in Figure 13.6.

Continuous Compounding of Interest

One question that arises naturally in the study of compound interest is, What happens to the accumulated amount over a fixed period of time if the interest is computed more and more frequently? Intuition suggests that the more often interest is compounded, the larger the accumulated amount will be. This is confirmed by the results of Example 3, Section 5.1, page 271, where we found that the accumulated amounts did in fact increase when we increased the number of conversion periods per year.

This leads us to another question: Does the accumulated amount approach a limit when the interest is computed more and more frequently over a fixed period of time? To answer this question, let's look again at the compound interest formula:

$$A = P\left(1 + \frac{r}{m}\right)^{mt} \tag{2}$$

Recall that m is the number of conversion periods per year. So to find an answer to our problem, we should let m approach infinity (get larger and larger) in (2). But first we rewrite this equation in the form

$$A = P\left[\left(1 + \frac{r}{m}\right)^{m}\right]^{t} \qquad \text{[Since } b^{xy} = (b^x)^y\text{]}$$

Now, letting $m \to \infty$, we find that

$$\lim_{m\to\infty}\left[P\left(1 + \frac{r}{m}\right)^{m}\right]^{t} = P\left[\lim_{m\to\infty}\left(1 + \frac{r}{m}\right)^{m}\right]^{t} \qquad \text{(Why?)}$$

Next, upon making the substitution $u = m/r$ and observing that $u \to \infty$ as $m \to \infty$, we reduce the foregoing expression to

$$P\left[\lim_{u\to\infty}\left(1 + \frac{1}{u}\right)^{ur}\right]^{t} = P\left[\lim_{u\to\infty}\left(1 + \frac{1}{u}\right)^{u}\right]^{rt}$$

But

$$\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u = e \qquad \text{[Using (1)]}$$

so

$$\lim_{m\to\infty} P\left[\left(1+\frac{r}{m}\right)^m\right]^t = Pe^{rt}$$

Our computations tell us that as the frequency with which interest is compounded increases without bound, the accumulated amount approaches Pe^{rt}. In this situation, we say that interest is *compounded continuously*. Let's summarize this important result.

Continuous Compound Interest Formula

$$A = Pe^{rt} \qquad \textbf{(3)}$$

where P = Principal
r = Annual interest rate compounded continuously
t = Time in years
A = Accumulated amount at the end of t years

Exploring with Technology

In the opening paragraph of Section 13.1, we pointed out that the accumulated amount of an account earning interest *compounded continuously* will eventually outgrow by far the accumulated amount of an account earning interest at the same nominal rate but earning simple interest. Illustrate this fact using the following example.

Suppose you deposit \$1000 in account I, earning interest at the rate of 10% per year compounded continuously so that the accumulated amount at the end of t years is $A_1(t) = 1000e^{0.1t}$. Suppose you also deposit \$1000 in account II, earning simple interest at the rate of 10% per year so that the accumulated amount at the end of t years is $A_2(t) = 1000(1 + 0.1t)$. Use a graphing utility to sketch the graphs of the functions A_1 and A_2 in the viewing rectangle $[0, 20] \times [0, 10{,}000]$ to see the accumulated amounts $A_1(t)$ and $A_2(t)$ over a 20-year period.

EXAMPLE 7 Find the accumulated amount after 3 years if \$1000 is invested at 8% per year compounded (a) daily (take the number of days in a year to be 365) and (b) continuously.

SOLUTION ✔

a. Using the compound interest formula with $P = 1000$, $r = 0.08$, $m = 365$, $t = 3$, and $n = (365)(3) = 1095$, we find

$$A = 1000\left(1+\frac{0.08}{365}\right)^{1095} \approx 1271.22 \qquad \left[A = P\left(1+\frac{r}{m}\right)^n\right]$$

or \$1271.22.

b. Here we use Formula (3) with $P = 1000$, $r = 0.08$, and $t = 3$, obtaining

$$A = 1000e^{(0.08)(3)}$$
$$\approx 1271.25 \qquad \text{(Using the } e^x \text{ key)}$$

or $1271.25. ■■■■

Observe that the accumulated amounts corresponding to interest compounded daily and interest compounded continuously differ by very little. The continuous compound interest formula is a very important tool in theoretical work in financial analysis.

If we solve Formula (3) for P, we obtain

$$P = Ae^{-rt} \qquad \textbf{(4)}$$

which gives the present value in terms of the future (accumulated) value for the case of continuous compounding.

EXAMPLE 8 The Blakely Investment Company owns an office building in the commercial district of a city. As a result of the continued success of an urban renewal program, local business is booming. The market value of Blakely's property is

$$V(t) = 300{,}000e^{\sqrt{t}/2}$$

where $V(t)$ is measured in dollars and t is the time in years from the present. If the expected rate of inflation is 9% compounded continuously for the next 10 years, find an expression for the present value $P(t)$ of the market price of the property valid for the next 10 years. Compute $P(7)$, $P(8)$, and $P(9)$, and interpret your results.

SOLUTION ✔ Using Formula (4) with $A = V(t)$ and $r = 0.09$, we find that the present value of the market price of the property t years from now is

$$P(t) = V(t)e^{-0.09t}$$
$$= 300{,}000e^{-0.09t+\sqrt{t}/2} \qquad (0 \le t \le 10)$$

Letting $t = 7$, 8, and 9, respectively, we find

$$P(7) = 300{,}000e^{-0.09(7)+\sqrt{7}/2} = 599{,}837 \quad \text{or} \quad \$599{,}837$$
$$P(8) = 300{,}000e^{-0.09(8)+\sqrt{8}/2} = 600{,}640 \quad \text{or} \quad \$600{,}640$$
$$P(9) = 300{,}000e^{-0.09(9)+\sqrt{9}/2} = 598{,}115 \quad \text{or} \quad \$598{,}115$$

From the results of these computations, we see that the present value of the property's market price seems to decrease after a certain period of growth. This suggests that there is an optimal time for the owners to sell. Later we will show that the highest present value of the property's market price is $600,779, which occurs at time $t = 7.72$ years. ■■■■

Self-Check Exercises 13.1

1. Solve the equation $2^{2x+1} \cdot 2^{-3} = 2^{x-1}$.
2. Sketch the graph of $y = e^{0.4x}$.
3. a. What is the accumulated amount after 5 yr if $10,000 is invested at an interest rate of 10%/year compounded continuously?
   b. Find the present value of $10,000 due in 5 yr at an interest rate of 10%/year compounded continuously.

Solutions to Self-Check Exercises 13.1 can be found on page 894.

Exploring with Technology

The effective rate of interest is given by

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

where the number of conversion periods per year is m. In Exercise 37 on page 893, you will be asked to show that the effective rate of interest r_{eff} corresponding to a nominal interest rate r per year compounded continuously is given by

$$\hat{r}_{\text{eff}} = e^r - 1$$

To obtain a visual confirmation of this result, consider the special case where $r = 0.1$ (10% per year).

1. Use a graphing utility to plot the graph of both

   $$y_1 = \left(1 + \frac{0.1}{x}\right)^x - 1 \qquad \text{and} \qquad y_2 = e^{0.1} - 1$$

   in the viewing rectangle $[0, 3] \times [0, 0.12]$.
2. Does your result seem to imply that

   $$\left(1 + \frac{r}{m}\right)^m - 1$$

   approaches

   $$\hat{r}_{\text{eff}} = e^r - 1$$

   as m increases without bound for the special case $r = 0.1$?

13.1 Exercises

In Exercises 1–6, evaluate each expression.

1. a. $4^{-3} \cdot 4^5$ b. $3^{-3} \cdot 3^6$
2. a. $(2^{-1})^3$ b. $(3^{-2})^3$
3. a. $9(9)^{-1/2}$ b. $5(5)^{-1/2}$
4. a. $\left[\left(-\frac{1}{2}\right)^3\right]^{-2}$ b. $\left[\left(-\frac{1}{3}\right)^2\right]^{-3}$

5. **a.** $\dfrac{(-3)^4(-3)^5}{(-3)^8}$ **b.** $\dfrac{(2^{-4})(2^6)}{2^{-1}}$

6. **a.** $3^{1/4} \cdot (9)^{-5/8}$ **b.** $2^{3/4} \cdot (4)^{-3/2}$

In Exercises 7–12, simplify each expression.

7. **a.** $(64x^9)^{1/3}$ **b.** $(25x^3y^4)^{1/2}$

8. **a.** $(2x^3)(-4x^{-2})$ **b.** $(4x^{-2})(-3x^5)$

9. **a.** $\dfrac{6a^{-5}}{3a^{-3}}$ **b.** $\dfrac{4b^{-4}}{12b^{-6}}$

10. **a.** $y^{-3/2}y^{5/3}$ **b.** $x^{-3/5}x^{8/3}$

11. **a.** $(2x^3y^2)^3$ **b.** $(4x^2y^2z^3)^2$

12. **a.** $\dfrac{5^0}{(2^{-3}x^{-3}y^2)^2}$ **b.** $\dfrac{(x+y)(x-y)}{(x-y)^0}$

In Exercises 13–20, solve the equation for *x*.

13. $6^{2x} = 6^4$

14. $5^{-x} = 5^3$

15. $3^{3x-4} = 3^5$

16. $10^{2x-1} = 10^{x+3}$

17. $(2.1)^{x+2} = (2.1)^5$

18. $(-1.3)^{x-2} = (-1.3)^{2x+1}$

19. $8^x = \left(\dfrac{1}{32}\right)^{x-2}$

20. $3^{x-x^2} = \dfrac{1}{9^x}$

In Exercises 21–29, sketch the graphs of the given functions on the same axes. A calculator is recommended for these exercises.

21. $y = 2^x$, $y = 3^x$, and $y = 4^x$

22. $y = \left(\dfrac{1}{2}\right)^x$, $y = \left(\dfrac{1}{3}\right)^x$, and $y = \left(\dfrac{1}{4}\right)^x$

23. $y = 2^{-x}$, $y = 3^{-x}$, and $y = 4^{-x}$

24. $y = 4^{0.5x}$ and $y = 4^{-0.5x}$

25. $y = 4^{0.5x}$, $y = 4^x$, and $y = 4^{2x}$

26. $y = e^x$, $y = 2e^x$, and $y = 3e^x$

27. $y = e^{0.5x}$, $y = e^x$, and $y = e^{1.5x}$

28. $y = e^{-0.5x}$, $y = e^{-x}$, and $y = e^{-1.5x}$

29. $y = 0.5e^{-x}$, $y = e^{-x}$, and $y = 2e^{-x}$

30. Find the accumulated amount after 4 yr if \$5000 is invested at 8%/year compounded continuously.

31. **INVESTMENT OPTIONS** Investment A offers a 10% return compounded semiannually, and investment B offers a 9.75% return compounded continuously. Which investment has a higher rate of return over a 4-yr period?

32. **PRESENT VALUE** Find the present value of \$59,673 due in 5 yr at an interest rate of 8%/year compounded continuously.

33. **SAVING FOR COLLEGE** Having received a large inheritance, a child's parents wish to establish a trust for the child's college education. If they need an estimated \$70,000 7 yr from now, how much should they set aside in trust now, if they invest the money at 10.5% compounded (a) quarterly? (b) continuously?

34. **EFFECT OF INFLATION ON SALARIES** Larry's current annual salary is \$25,000. Ten years from now, how much will he need to earn in order to retain his present purchasing power if the rate of inflation over that period is 6%/year? Assume that inflation is continuously compounded.

35. **PENSIONS** Carmen, who is now 50 years old, is employed by a firm that guarantees her a pension of \$40,000/year at age 65. What is the present value of her first year's pension if inflation over the next 15 years is (a) 6%? (b) 8%? (c) 12%? Assume that inflation is continuously compounded.

36. **REAL ESTATE INVESTMENTS** An investor purchased a piece of waterfront property. Because of the development of a marina in the vicinity, the market value of the property is expected to increase according to the rule

$$V(t) = 80{,}000e^{\sqrt{t}/2}$$

where $V(t)$ is measured in dollars and t is the time in years from the present. If the rate of inflation is expected to be 9% compounded continuously for the next 8 yr, find an expression for the present value $P(t)$ of the property's market price valid for the next 8 yr. What is $P(t)$ expected to be in 4 yr?

37. Show that the effective rate of interest $\hat{r}_{\text{eff}}$ that corresponds to a nominal interest rate r per year compounded continuously is given by

$$\hat{r}_{\text{eff}} = e^r - 1$$

Hint: From Formula (4) in Section 5.1, page 273, we see that the effective rate $\hat{r}_{\text{eff}}$ corresponding to a nominal interest rate r per year compounded m times a year is given by

$$\hat{r}_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

Let m tend to infinity in this expression.

38. Refer to Exercise 37. Find the effective rate of interest that corresponds to a nominal rate of 10%/year compounded (a) quarterly, (b) monthly, and (c) continuously.

39. Investment Analysis Refer to Exercise 37. Bank A pays interest on deposits at a 7% annual rate compounded quarterly, and bank B pays interest on deposits at a $7\frac{1}{8}$% annual rate compounded continuously. Which bank has the higher effective rate of interest?

Solutions to Self-Check Exercises 13.1

1.

$$2^{2x+1} \cdot 2^{-3} = 2^{x-1}$$

$$\frac{2^{2x+1}}{2^{x-1}} \cdot 2^{-3} = 1 \qquad \text{(Dividing both sides by } 2^{x-1}\text{)}$$

$$2^{(2x+1)-(x-1)-3} = 1$$

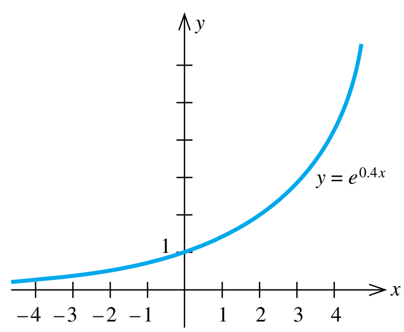
$$2^{x-1} = 1$$

This is true if and only if $x - 1 = 0$ or $x = 1$.

2. We first construct the following table of values:

x	−3	−2	−1	0	1	2	3	4
$y = e^{0.4x}$	0.3	0.5	0.7	1	1.5	2.2	3.3	5

Next, we plot these points and join them by a smooth curve to obtain the graph of f shown in the accompanying figure:



3. a. Using Formula (3) with $P = 10{,}000$, $r = 0.1$, and $t = 5$, we find that the required accumulated amount is given by

$$A = 10{,}000e^{(0.1)(5)}$$
$$= 16{,}487.21$$

or $16,487.21.

b. Using Formula (4) with $A = 10{,}000$, $r = 0.1$, and $t = 5$, we see that the required present value is given by

$$P = 10{,}000e^{-(0.1)(5)}$$
$$= 6065.31$$

or $6065.31.

(continued on p. 898)

Portfolio

MISATO NAKAZAKI

TITLE: Assistant Vice President
INSTITUTION: A large investment corporation

In the securities industry, buying and selling stocks and bonds has always required a mastery of concepts and formulas that outsiders find confusing. As a bond seller, Misato Nakazaki routinely uses terms such as *issue, maturity, current yield, callable* and *convertible bonds,* and so on.

These terms, however, are easily defined. When corporations issue bonds, they are borrowing money at a fixed rate of interest. The bonds are scheduled to mature—to be paid back—on a specific date as much as 30 years into the future. Callable bonds allow the issuer to pay off the loans prior to their expected maturity, reducing overall interest payments. In its simplest terms, current yield is the price of a bond multiplied by the interest rate at which the bond is issued. For example, a bond with a face value of $1000 and an interest rate of 10% yields $100 per year in interest payments. When that same bond is resold at a premium on the secondary market for $1200, its current yield nets only an 8.3% rate of return based on the higher purchase price.

Bonds attract investors for many reasons. A key variable is the sensitivity of the bond's price to future changes in interest rates. If investors get locked into a low-paying bond when future bonds pay higher yields, they lose money. Nakazaki stresses that "no one knows for sure what rates will be over time." Employing differentials allows her to calculate interest-rate sensitivity for clients as they ponder purchase decisions.

Computerized formulas, "whose basis is calculus," says Nakazaki, help her factor the endless stream of numbers flowing across her desk.

On a typical day, Nakazaki might be given a bid on "10 million, GMAC, 8.5%, January 2003." Translation: Her customer wants her to buy General Motors Acceptance Corporation bonds with a face value of $10 million and an interest rate of 8.5%, maturing in January 2003.

After she calls her firm's trader to find out the yield on the bond in question, Nakazaki enters the price and other variables, such as the interest rate and date of maturity, and the computer prints out the answers. Nakazaki can then relay to her client the bond's current yield, accrued interest, and so on. In Nakazaki's rapid-fire work environment, such speed is essential. Nakazaki cautions that "computer users have to understand what's behind the formulas." The software "relies on the basics of calculus. If people don't understand the formula, it's useless for them to use the calculations."

With an MBA from New York University, Nakazaki typifies the younger generation of Japanese women who have chosen to succeed in the business world. Since earning her degree, she has sold bonds for a global securities firm in New York City.

Nakazaki's client list reads like a *who's who* of the leading Japanese banks, insurance companies, mutual funds, and corporations. As institutional buyers, her clients purchase large blocks of American corporate bonds and mortgage-backed securities such as Ginnie Maes.

Using Technology

Although the proof is outside the scope of this book, it can be proved that an exponential function of the form $f(x) = b^x$, where $b > 1$, will ultimately grow faster than the power function $g(x) = x^n$ for *any* positive real number n. To give a visual demonstration of this result for the special case of the exponential function $f(x) = e^x$, we can use a graphing calculator to plot the graphs of both f and g (for selected values of n) on the same set of axes in an appropriate viewing rectangle and observe that the graph of f ultimately lies above that of g.

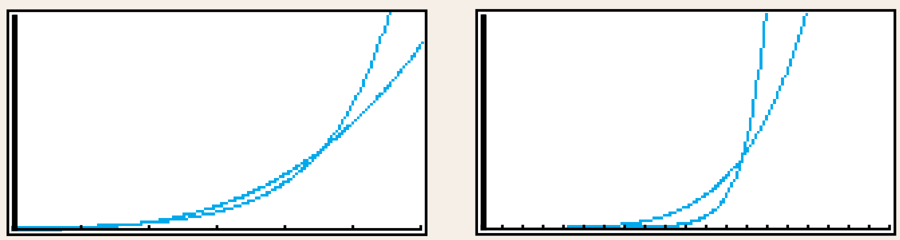
EXAMPLE 1 Use a graphing utility to plot the graphs of (a) $f(x) = e^x$ and $g(x) = x^3$ on the same set of axes in the viewing rectangle $[0, 6] \times [0, 250]$ and (b) $f(x) = e^x$ and $g(x) = x^5$ in the viewing rectangle $[0, 20] \times [0, 1{,}000{,}000]$.

SOLUTION ✔

a. The graphs of $f(x) = e^x$ and $g(x) = x^3$ in the viewing rectangle $[0, 6] \times [0, 250]$ are shown in Figure T1a.

b. The graphs of $f(x) = e^x$ and $g(x) = x^5$ in the viewing rectangle $[0, 20] \times [0, 1{,}000{,}000]$ are shown in Figure T1b.

FIGURE T1



(a) The graphs of $f(x) = e^x$ and $g(x) = x^3$ in the viewing rectangle $[0, 6] \times [0, 250]$

(b) The graphs of $f(x) = e^x$ and $g(x) = x^5$ in the viewing rectangle $[0, 20] \times [0, 1{,}000{,}000]$

In the exercises that follow, you are asked to use a graphing utility to reveal the properties of exponential functions.

Exercises

In Exercises 1 and 2, use a graphing utility to plot the graphs of the functions *f* and *g* on the same set of axes in the specified viewing rectangle.

1. $f(x) = e^x$ and $g(x) = x^2$; $[0, 4] \times [0, 30]$
2. $f(x) = e^x$ and $g(x) = x^4$; $[0, 15] \times [0, 20{,}000]$

In Exercises 3 and 4, use a graphing utility to plot the graphs of the functions *f* and *g* on the same set of axes in an appropriate viewing rectangle to demonstrate that *f* ultimately grows faster than *g*. (*Note:* Your answer will not be unique.)

3. $f(x) = 2^x$ and $g(x) = x^{2.5}$
4. $f(x) = 3^x$ and $g(x) = x^3$
5. Use a graphing utility to plot the graphs of $f(x) = 2^x$, $g(x) = 3^x$, and $h(x) = 4^x$ on the same set of axes in the viewing rectangle $[0, 5] \times [0, 100]$. Comment on the relationship between the base b and the growth of the function $f(x) = b^x$.
6. Use a graphing utility to plot the graphs of $f(x) = (1/2)^x$, $g(x) = (1/3)^x$, and $h(x) = (1/4)^x$ on the same set of axes in the viewing rectangle $[0, 4] \times [0, 1]$. Comment on the relationship between the base b and the growth of the function $f(x) = b^x$.
7. Use a graphing utility to plot the graphs of $f(x) = e^x$, $g(x) = 2e^x$, and $h(x) = 3e^x$ on the same set of axes in the viewing rectangle $[-3, 3] \times [0, 10]$. Comment on the role played by the constant k in the graph of $f(x) = ke^x$.
8. Use a graphing utility to plot the graphs of $f(x) = -e^x$, $g(x) = -2e^x$, and $h(x) = -3e^x$ on the same set of axes in the viewing rectangle $[-3, 3] \times [-10, 0]$. Comment on the role played by the constant k in the graph of $f(x) = ke^x$.
9. Use a graphing utility to plot the graphs of $f(x) = e^{0.5x}$, $g(x) = e^x$, and $h(x) = e^{1.5x}$ on the same set of axes in the viewing rectangle $[-2, 2] \times [0, 4]$. Comment on the role played by the constant k in the graph of $f(x) = e^{kx}$.
10. Use a graphing utility to plot the graphs of $f(x) = e^{-0.5x}$, $g(x) = e^{-x}$, and $h(x) = e^{-1.5x}$ on the same set of axes in the viewing rectangle $[-2, 2] \times [0, 4]$. Comment on the role played by the constant k in the graph of $f(x) = e^{kx}$.

13.2 Logarithmic Functions

LOGARITHMS

You are already familiar with exponential equations of the form

$$b^y = x \qquad (b > 0, b \neq 1)$$

where the variable x is expressed in terms of a real number b and a variable y. But what about solving this same equation for y? You may recall from your study of algebra that the number y is called the **logarithm of x to the base b** and is denoted by **$\log_b x$.** It is the exponent to which the base b must be raised in order to obtain the number x.

Logarithm of x to the Base b

$$y = \log_b x \qquad \text{if and only if} \qquad x = b^y \qquad (x > 0)$$

Observe that the logarithm $\log_b x$ is defined only for positive values of x.

EXAMPLE 1

a. $\log_{10} 100 = 2$ since $100 = 10^2$
b. $\log_5 125 = 3$ since $125 = 5^3$
c. $\log_3 \dfrac{1}{27} = -3$ since $\dfrac{1}{27} = \dfrac{1}{3^3} = 3^{-3}$
d. $\log_{20} 20 = 1$ since $20 = 20^1$

■■■■

EXAMPLE 2 Solve each of the following equations for x.

a. $\log_3 x = 4$ **b.** $\log_{16} 4 = x$ **c.** $\log_x 8 = 3$

SOLUTION ✔

a. By definition, $\log_3 x = 4$ implies $x = 3^4 = 81$.
b. $\log_{16} 4 = x$ is equivalent to $4 = 16^x = (4^2)^x = 4^{2x}$, or $4^1 = 4^{2x}$, from which we deduce that

$$2x = 1 \qquad (b^m = b^n \Rightarrow m = n)$$
$$x = \frac{1}{2}$$

c. Referring once again to the definition, we see that the equation $\log_x 8 = 3$ is equivalent to

$$8 = (2^3) = x^3$$
$$x = 2 \qquad (a^m = b^m \Rightarrow a = b)$$

■■■■

The two widely used systems of logarithms are the system of **common logarithms,** which uses the number 10 as the base, and the system of **natural logarithms,** which uses the irrational number $e = 2.71828\ldots$ as the base. Also, it is standard practice to write **log** for $\log_{10}$ and **ln** for $\log_e$.

Logarithmic Notation

$$\log x = \log_{10} x \qquad \text{(Common logarithm)}$$
$$\ln x = \log_e x \qquad \text{(Natural logarithm)}$$

The system of natural logarithms is widely used in theoretical work. Using natural logarithms rather than logarithms to other bases often leads to simpler expressions.

Laws of Logarithms

Computations involving logarithms are facilitated by the following **laws of logarithms.**

Laws of Logarithms

If m and n are positive numbers, then

1. $\log_b mn = \log_b m + \log_b n$
2. $\log_b \dfrac{m}{n} = \log_b m - \log_b n$
3. $\log_b m^n = n \log_b m$
4. $\log_b 1 = 0$
5. $\log_b b = 1$

Do not confuse the expression $\log m/n$ (Law 2) with the expression $\log m/\log n$. For example,

$$\log \frac{100}{10} = \log 100 - \log 10 = 2 - 1 = 1 \neq \frac{\log 100}{\log 10} = \frac{2}{1} = 2$$

You will be asked to prove these laws in Exercises 53–55. Their derivations are based on the definition of a logarithm and the corresponding laws of exponents. The following examples illustrate the properties of logarithms.

EXAMPLE 3

a. $\log(2 \cdot 3) = \log 2 + \log 3$ **b.** $\ln \frac{5}{3} = \ln 5 - \ln 3$

c. $\log \sqrt{7} = \log 7^{1/2} = \frac{1}{2} \log 7$ **d.** $\log_5 1 = 0$

e. $\log_{45} 45 = 1$ ■■■■

EXAMPLE 4

Given that $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, and $\log 5 \approx 0.6990$, use the laws of logarithms to find

a. $\log 15$ **b.** $\log 7.5$ **c.** $\log 81$ **d.** $\log 50$

SOLUTION ✔

a. Note that $15 = 3 \cdot 5$, so by Law 1 for logarithms,

$$\begin{aligned} \log 15 &= \log 3 \cdot 5 \\ &= \log 3 + \log 5 \\ &\approx 0.4771 + 0.6990 \\ &= 1.1761 \end{aligned}$$

b. Observing that $7.5 = 15/2 = (3 \cdot 5)/2$, we apply Laws 1 and 2, obtaining

$$\begin{aligned} \log 7.5 &= \log \frac{(3)(5)}{2} \\ &= \log 3 + \log 5 - \log 2 \\ &\approx 0.4771 + 0.6990 - 0.3010 \\ &= 0.8751 \end{aligned}$$

c. Since $81 = 3^4$, we apply Law 3 to obtain

$$\begin{aligned} \log 81 &= \log 3^4 \\ &= 4 \log 3 \\ &\approx 4(0.4771) \\ &= 1.9084 \end{aligned}$$

d. We write $50 = 5 \cdot 10$ and find

$$\begin{aligned} \log 50 &= \log(5)(10) \\ &= \log 5 + \log 10 \\ &\approx 0.6990 + 1 && \text{(Using Law 5)} \\ &= 1.6990 \end{aligned}$$

■■■■

EXAMPLE 5

Expand and simplify the following expressions:

a. $\log_3 x^2y^3$ **b.** $\log_2 \dfrac{x^2+1}{2^x}$ **c.** $\ln \dfrac{x^2\sqrt{x^2-1}}{e^x}$

SOLUTION ✔

a.
$$\begin{aligned} \log_3 x^2y^3 &= \log_3 x^2 + \log_3 y^3 && \text{(Law 1)} \\ &= 2\log_3 x + 3\log_3 y && \text{(Law 3)} \end{aligned}$$

b.
$$\begin{aligned} \log_2 \frac{x^2+1}{2^x} &= \log_2(x^2+1) - \log_2 2^x && \text{(Law 2)} \\ &= \log_2(x^2+1) - x\log_2 2 && \text{(Law 3)} \\ &= \log_2(x^2+1) - x && \text{(Law 5)} \end{aligned}$$

c.
$$\begin{aligned} \ln \frac{x^2\sqrt{x^2-1}}{e^x} &= \ln \frac{x^2(x^2-1)^{1/2}}{e^x} && \text{(Rewriting)} \\ &= \ln x^2 + \ln(x^2-1)^{1/2} - \ln e^x && \text{(Laws 1 and 2)} \\ &= 2\ln x + \frac{1}{2}\ln(x^2-1) - x\ln e && \text{(Law 3)} \\ &= 2\ln x + \frac{1}{2}\ln(x^2-1) - x && \text{(Law 5)} \end{aligned}$$

■■■■

LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

The definition of the logarithm implies that if b and n are positive numbers and b is different from 1, then the expression $\log_b n$ is a real number. This enables us to define a logarithmic function as follows:

Logarithmic Function

The function defined by

$$f(x) = \log_b x \qquad (b > 0, b \neq 1)$$

is called the **logarithmic function with base b.** The domain of f is the set of all positive numbers.

One easy way to obtain the graph of the logarithmic function $y = \log_b x$ is to construct a table of values of the logarithm (base b). However, another method—and a more instructive one—is based on exploiting the intimate relationship between logarithmic and exponential functions.

If a point (u, v) lies on the graph of $y = \log_b x$, then

$$v = \log_b u$$

But we can also write this equation in exponential form as

$$u = b^v$$

So the point (v, u) also lies on the graph of the function $y = b^x$. Let's look at the relationship between the points (u, v) and (v, u) and the line $y = x$ (Figure 13.7). If we think of the line $y = x$ as a mirror, then the point (v, u) is the mirror reflection of the point (u, v). Similarly, the point (u, v) is a mirror reflection of the point (v, u). We can take advantage of this relationship to help us draw the graph of logarithmic functions. For example, if we wish to draw the graph of $y = \log_b x$, where $b > 1$, then we need only draw the mirror reflection of the graph of $y = b^x$ with respect to the line $y = x$ (Figure 13.8).

FIGURE 13.7
The points (u, v) and (v, u) are mirror reflections of each other.

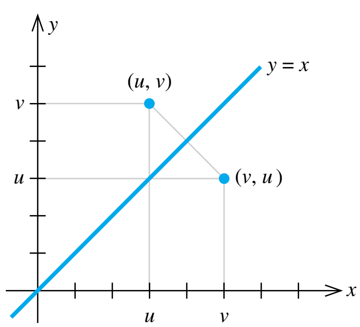
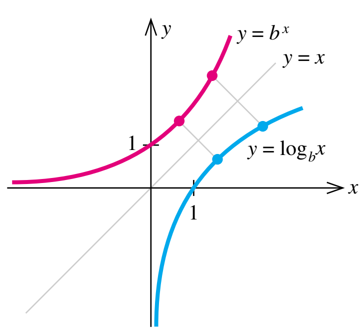


FIGURE 13.8
The graphs of $y = b^x$ and $y = \log_b x$ are mirror reflections of each other.



You may discover the following properties of the logarithmic function by taking the reflection of the graph of an appropriate exponential function (Exercises 31 and 32).

Properties of the Logarithmic Function

The logarithmic function $y = \log_b x$ ($b > 0$, $b \neq 1$) has the following properties:

1. Its domain is $(0, \infty)$.
2. Its range is $(-\infty, \infty)$.
3. Its graph passes through the point $(1, 0)$.
4. It is continuous on $(0, \infty)$.
5. It is increasing on $(0, \infty)$ if $b > 1$ and decreasing on $(0, \infty)$ if $b < 1$.

EXAMPLE 6 Sketch the graph of the function $y = \ln x$.

SOLUTION ✔ We first sketch the graph of $y = e^x$. Then, the required graph is obtained by tracing the mirror reflection of the graph of $y = e^x$ with respect to the line $y = x$ (Figure 13.9).

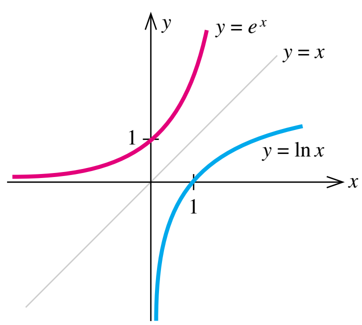


FIGURE 13.9
The graph of $y = \ln x$ is the mirror reflection of the graph of $y = e^x$.

PROPERTIES RELATING THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We made use of the relationship that exists between the exponential function $f(x) = e^x$ and the logarithmic function $g(x) = \ln x$ when we sketched the graph of g in Example 6. This relationship is further described by the following properties, which are an immediate consequence of the definition of the logarithm of a number.

Properties Relating e^x and $\ln x$

$$e^{\ln x} = x \qquad (x > 0) \qquad \textbf{(5)}$$
$$\ln e^x = x \qquad (\text{for any real number } x) \qquad \textbf{(6)}$$

(Try to verify these properties.)

From Properties 5 and 6, we conclude that the composite function

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= e^{\ln x} = x \\ (g \circ f)(x) &= g[f(x)] \\ &= \ln e^x = x\end{aligned}$$

Thus,

$$f[g(x)] = g[f(x)]$$
$$= x$$

Any two functions f and g that satisfy this relationship are said to be **inverses** of each other. Note that the function f undoes what the function g does, and vice versa, so the composition of the two functions in any order results in the identity function $F(x) = x$.

The relationships expressed in Equations (5) and (6) are useful in solving equations that involve exponentials and logarithms.

Exploring with Technology

You can demonstrate the validity of Properties 5 and 6, which state that the exponential function $f(x) = e^x$ and the logarithmic function $g(x) = \ln x$ are inverses of each other as follows:

1. Sketch the graph of $(f \circ g)(x) = e^{\ln x}$, using the viewing rectangle $[0, 10] \times [0, 10]$. Interpret the result.
2. Sketch the graph of $(g \circ f)(x) = \ln e^x$, using the standard viewing rectangle. Interpret the result.

EXAMPLE 7 Solve the equation $2e^{x+2} = 5$.

SOLUTION ✓ We first divide both sides of the equation by 2 to obtain

$$e^{x+2} = \frac{5}{2} = 2.5$$

Next, taking the natural logarithm of each side of the equation and using Equation (6), we have

$$\begin{aligned} \ln e^{x+2} &= \ln 2.5 \\ x + 2 &= \ln 2.5 \\ x &= -2 + \ln 2.5 \\ &\approx -1.08 \end{aligned}$$

■■■■

EXAMPLE 8 Solve the equation $5 \ln x + 3 = 0$.

SOLUTION ✓ Adding -3 to both sides of the equation leads to

$$\begin{aligned} 5 \ln x &= -3 \\ \ln x &= -\frac{3}{5} = -0.6 \end{aligned}$$

and so

$$e^{\ln x} = e^{-0.6}$$

Using equation (5), we conclude that

$$x = e^{-0.6}$$
$$\approx 0.55$$

Group Discussion

Consider the equation $y = y_0 b^{kx}$, where y_0 and k are positive constants and $b > 0$, $b \neq 1$. Suppose we want to express y in the form $y = y_0 e^{px}$. Use the laws of logarithms to show that $p = k \ln b$ and hence that $y = y_0 e^{(k \ln b)x}$ is an alternative form of $y = y_0 b^{kx}$ using the base e.

Self-Check Exercises 13.2

1. Sketch the graph of $y = 3^x$ and $y = \log_3 x$ on the same set of axes.
2. Solve the equation $3e^{x+1} - 2 = 4$.

Solutions to Self-Check Exercises 13.2 can be found on page 906.

13.2 Exercises

In Exercises 1–10, express the given equation in logarithmic form.

1. $2^6 = 64$
2. $3^5 = 243$
3. $3^{-2} = \frac{1}{9}$
4. $5^{-3} = \frac{1}{125}$
5. $\left(\frac{1}{3}\right)^1 = \frac{1}{3}$
6. $\left(\frac{1}{2}\right)^{-4} = 16$
7. $32^{3/5} = 8$
8. $81^{3/4} = 27$
9. $10^{-3} = 0.001$
10. $16^{-1/4} = 0.5$

In Exercises 11–16, use the facts that log 3 = 0.4771 and log 4 = 0.6021 to find the value of the given logarithm.

11. $\log 12$
12. $\log \frac{3}{4}$
13. $\log 16$
14. $\log \sqrt{3}$
15. $\log 48$
16. $\log \frac{1}{300}$

In Exercises 17–26, use the laws of logarithms to simplify the given expression.

17. $\log x(x+1)^4$
18. $\log x(x^2+1)^{-1/2}$
19. $\log \frac{\sqrt{x+1}}{x^2+1}$
20. $\ln \frac{e^x}{1+e^x}$
21. $\ln xe^{-x^2}$
22. $\ln x(x+1)(x+2)$
23. $\ln \frac{x^{1/2}}{x^2\sqrt{1+x^2}}$
24. $\ln \frac{x^2}{\sqrt{x}(1+x)^2}$
25. $\ln x^x$
26. $\ln x^{x^2+1}$

In Exercises 27–30, sketch the graph of the given equation.

27. $y = \log_3 x$
28. $y = \log_{1/3} x$
29. $y = \ln 2x$
30. $y = \ln \frac{1}{2} x$

In Exercises 31 and 32, sketch the graphs of the given equations on the same coordinate axes.

31. $y = 2^x$ and $y = \log_2 x$

32. $y = e^{3x}$ and $y = \ln 3x$

In Exercises 33–42, use logarithms to solve the given equation for *t*.

33. $e^{0.4t} = 8$

34. $\frac{1}{3} e^{-3t} = 0.9$

35. $5e^{-2t} = 6$

36. $4e^{t-1} = 4$

37. $2e^{-0.2t} - 4 = 6$

38. $12 - e^{0.4t} = 3$

39. $\dfrac{50}{1 + 4e^{0.2t}} = 20$

40. $\dfrac{200}{1 + 3e^{-0.3t}} = 100$

41. $A = Be^{-t/2}$

42. $\dfrac{A}{1 + Be^{t/2}} = C$

43. Blood Pressure A normal child's systolic blood pressure may be approximated by the function

$$p(x) = m(\ln x) + b$$

where $p(x)$ is measured in millimeters of mercury, x is measured in pounds, and m and b are constants. Given that $m = 19.4$ and $b = 18$, determine the systolic blood pressure of a child who weighs 92 lb.

44. Magnitude of Earthquakes On the Richter scale, the magnitude R of an earthquake is given by the formula

$$R = \log \frac{I}{I_0}$$

where I is the intensity of the earthquake being measured and I_0 is the standard reference intensity.

a. Express the intensity I of an earthquake of magnitude $R = 5$ in terms of the standard intensity I_0.

b. Express the intensity I of an earthquake of magnitude $R = 8$ in terms of the standard intensity I_0. How many times greater is the intensity of an earthquake of magnitude 8 than one of magnitude 5?

c. In modern times the greatest loss of life attributable to an earthquake occurred in eastern China in 1976. Known as the Tangshan earthquake, it registered 8.2 on the Richter scale. How does the intensity of this earthquake compare with the intensity of an earthquake of magnitude $R = 5$?

45. Sound Intensity The relative loudness of a sound D of intensity I is measured in decibels (db), where

$$D = 10 \log \frac{I}{I_0}$$

and I_0 is the standard threshold of audibility.

a. Express the intensity I of a 30-db sound (the sound level of normal conversation) in terms of I_0.

b. Determine how many times greater the intensity of an 80-db sound (rock music) is than that of a 30-db sound.

c. Prolonged noise above 150 db causes immediate and permanent deafness. How does the intensity of a 150-db sound compare with the intensity of an 80-db sound?

46. Barometric Pressure Halley's law states that the barometric pressure (in inches of mercury) at an altitude of x mi above sea level is approximately given by the equation

$$p(x) = 29.92e^{-0.2x} \qquad (x \geq 0)$$

If the barometric pressure as measured by a hot-air balloonist is 20 in. of mercury, what is the balloonist's altitude?

47. Forensic Science Forensic scientists use the following law to determine the time of death of accident or murder victims. If T denotes the temperature of a body t hr after death, then

$$T = T_0 + (T_1 - T_0)(0.97)^t$$

where T_0 is the air temperature and T_1 is the body temperature at the time of death. John Doe was found murdered at midnight in his house, when the room temperature was 70°F and his body temperature was 80°F. When was he killed? Assume that the normal body temperature is 98.6°F.

In Exercises 48–51, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

48. $(\ln x)^3 = 3 \ln x$ for all x in $(0, \infty)$.

49. $\ln a - \ln b = \ln(a - b)$ for all positive real numbers a and b.

50. The function $f(x) = 1/\ln x$ is continuous on $(1, \infty)$.

51. The function $f(x) = \ln |x|$ is continuous for all $x \neq 0$.

52. **a.** Given that $2^x = e^{kx}$, find k.
b. Show that, in general, if b is a nonnegative real number, then any equation of the form $y = b^x$ may be written in the form $y = e^{kx}$, for some real number k.

53. Use the definition of a logarithm to prove:
a. $\log_b mn = \log_b m + \log_b n$
b. $\log_b \frac{m}{n} = \log_b m - \log_b n$

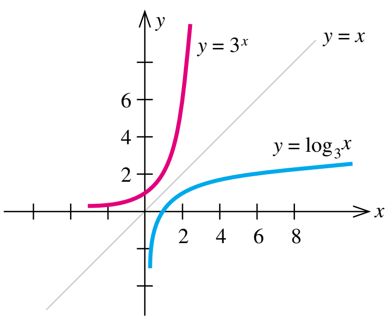
Hint: Let $\log_b m = p$ and $\log_b n = q$. Then, $b^p = m$ and $b^q = n$.

54. Use the definition of a logarithm to prove

$$\log_b m^n = n \log_b m$$

55. Use the definition of a logarithm to prove:
a. $\log_b 1 = 0$
b. $\log_b b = 1$

SOLUTIONS TO SELF-CHECK EXERCISES 13.2



1. First, sketch the graph of $y = 3^x$ with the help of the following table of values:

x	-3	-2	-1	0	1	2	3
$y = 3^x$	1/27	1/9	1/3	0	3	9	27

Next, take the mirror reflection of this graph with respect to the line $y = x$ to obtain the graph of $y = \log_3 x$.

2.

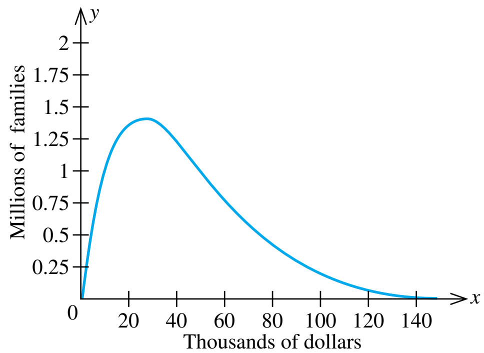
$$\begin{aligned} 3e^{x+1} - 2 &= 4 \\ 3e^{x+1} &= 6 \\ e^{x+1} &= 2 \\ \ln e^{x+1} &= \ln 2 \\ (x+1)\ln e &= \ln 2 \qquad \text{(Law 3)} \\ x + 1 &= \ln 2 \qquad \text{(Law 5)} \\ x &= \ln 2 - 1 \\ &\approx -0.3069 \end{aligned}$$

13.3 Differentiation of Exponential Functions

THE DERIVATIVE OF THE EXPONENTIAL FUNCTION

To study the effects of budget deficit-reduction plans at different income levels, it is important to know the income distribution of American families. Based on data from the House Budget Committee, the House Ways and Means Committee, and the U.S. Census Bureau, the graph of f shown in Figure 13.10 gives the number of American families y (in millions) as a function of their annual income x (in thousands of dollars) in 1990.

FIGURE 13.10
The graph of f shows the number of families versus their annual income.



Source: House Budget Committee, House Ways and Means Committee, and U.S. Census Bureau

Observe that the graph of f rises very quickly and then tapers off. From the graph of f, you can see that the bulk of American families earned less than \$100,000 per year. In fact, 95% of U.S. families earned less than \$102,358 per year in 1990. (We will refer to this model again in Using Technology at the end of this section.)

To analyze mathematical models involving exponential and logarithmic functions in greater detail, we need to develop rules for computing the derivative of these functions. We begin by looking at the rule for computing the derivative of the exponential function.

Rule 1: Derivative of the Exponential Function

$$\frac{d}{dx}e^x = e^x$$

Thus, the derivative of the exponential function with base e is equal to the function itself. To demonstrate the validity of this rule, we compute

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h} && \text{(Writing } e^{x+h} = e^xe^h \text{ and factoring)} \\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h} && \text{(Why?)} \end{aligned}$$

To evaluate

$$\lim_{h \to 0} \frac{e^h - 1}{h}$$

let's refer to Table 13.2, which is constructed with the aid of a calculator. From the table, we see that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

(Although a rigorous proof of this fact is possible, it is beyond the scope of this book. Also see Example 1, Using Technology, page 916.) Using this result, we conclude that

$$f'(x) = e^x \cdot 1 = e^x$$

as we set out to show.

Table 13.2

h	0.1	0.01	0.001	−0.1	−0.01	−0.001
$\frac{e^h - 1}{h}$	1.0517	1.0050	1.0005	0.9516	0.9950	0.9995

EXAMPLE 1 Compute the derivative of each of the following functions:

a. $f(x) = x^2e^x$ **b.** $g(t) = (e^t + 2)^{3/2}$

SOLUTION **a.** The product rule gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^2e^x) \\ &= x^2\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2) \\ &= x^2e^x + e^x(2x) \\ &= xe^x(x + 2) \end{aligned}$$

b. Using the general power rule, we find

$$\begin{aligned} g'(t) &= \frac{3}{2}(e^t + 2)^{1/2}\frac{d}{dt}(e^t + 2) \\ &= \frac{3}{2}(e^t + 2)^{1/2}e^t = \frac{3}{2}e^t(e^t + 2)^{1/2} \end{aligned}$$

Exploring with Technology

Consider the exponential function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

1. Use the definition of the derivative of a function to show that

$$f'(x) = b^x \cdot \lim_{h \to 0} \frac{b^h - 1}{h}$$

2. Use the result of part 1 to show that

$$\frac{d}{dx}(2^x) = 2^x \cdot \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$\frac{d}{dx}(3^x) = 3^x \cdot \lim_{h \to 0} \frac{3^h - 1}{h}$$

3. Use the technique in Using Technology, page 916, to show that (to two decimal places)

$$\lim_{h \to 0} \frac{2^h - 1}{h} = 0.69 \qquad \text{and} \qquad \lim_{h \to 0} \frac{3^h - 1}{h} = 1.10$$

4. Conclude from the results of parts 2 and 3 that

$$\frac{d}{dx}(2^x) \approx (0.69)2^x \qquad \text{and} \qquad \frac{d}{dx}(3^x) \approx (1.10)3^x$$

Thus,

$$\frac{d}{dx}(b^x) = k \cdot b^x$$

where k is an appropriate constant.

5. The results of part 4 suggest that, for convenience, we pick the base b, where $2 < b < 3$, so that $k = 1$. This value of b is $e \approx 2.718281828\ldots$. Thus,

$$\frac{d}{dx}(e^x) = e^x$$

This is why we prefer to work with the exponential function $f(x) = e^x$.

Applying the Chain Rule to Exponential Functions

To enlarge the class of exponential functions to be differentiated, we appeal to the chain rule to obtain the following rule for differentiating composite functions of the form $h(x) = e^{f(x)}$. An example of such a function is $h(x) = e^{x^2-2x}$. Here, $f(x) = x^2 - 2x$.

Rule 2: Chain Rule for Exponential Functions

If $f(x)$ is a differentiable function, then

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)}f'(x)$$

To see this, observe that if $h(x) = g[f(x)]$, where $g(x) = e^x$, then by virtue of the chain rule,

$$h'(x) = g'(f(x))f'(x) = e^{f(x)}f'(x)$$

since $g'(x) = e^x$.

As an aid to remembering the chain rule for exponential functions, observe that it has the following form:

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)} \cdot \text{derivative of exponent}$$

↑Same↑

EXAMPLE 2 Find the derivative of each of the following functions.

a. $f(x) = e^{2x}$ **b.** $y = e^{-3x}$ **c.** $g(t) = e^{2t^2+t}$

SOLUTION ✔

a. $f'(x) = e^{2x}\frac{d}{dx}(2x) = e^{2x} \cdot 2 = 2e^{2x}$

b. $\frac{dy}{dx} = e^{-3x}\frac{d}{dx}(-3x) = -3e^{-3x}$

c. $g'(t) = e^{2t^2+t} \cdot \frac{d}{dt}(2t^2 + t) = (4t + 1)e^{2t^2+t}$

■■■■

EXAMPLE 3 Differentiate the function $y = xe^{-2x}$.

SOLUTION ✔ Using the product rule, followed by the chain rule, we find

$$\begin{aligned}\frac{dy}{dx} &= x\frac{d}{dx}e^{-2x} + e^{-2x}\frac{d}{dx}(x)\\ &= xe^{-2x}\frac{d}{dx}(-2x) + e^{-2x} \quad \text{(Using the chain rule on the first term)}\\ &= -2xe^{-2x} + e^{-2x}\\ &= e^{-2x}(1 - 2x)\end{aligned}$$

■■■■

EXAMPLE 4 Differentiate the function $g(t) = \dfrac{e^t}{e^t + e^{-t}}$.

SOLUTION ✓ Using the quotient rule, followed by the chain rule, we find

$$\begin{aligned} g'(t) &= \frac{(e^t + e^{-t})\dfrac{d}{dt}(e^t) - e^t\dfrac{d}{dt}(e^t + e^{-t})}{(e^t + e^{-t})^2} \\ &= \frac{(e^t + e^{-t})e^t - e^t(e^t - e^{-t})}{(e^t + e^{-t})^2} \\ &= \frac{e^{2t} + 1 - e^{2t} + 1}{(e^t + e^{-t})^2} \qquad (e^0 = 1) \\ &= \frac{2}{(e^t + e^{-t})^2} \end{aligned}$$

■■■■

EXAMPLE 5 In Section 13.5 we will discuss some practical applications of the exponential function

$$Q(t) = Q_0e^{kt}$$

where Q_0 and k are positive constants and $t \in [0, \infty)$. A quantity $Q(t)$ growing according to this law experiences exponential growth. Show that for a quantity $Q(t)$ experiencing exponential growth, the rate of growth of the quantity $Q'(t)$ at any time t is directly proportional to the amount of the quantity present.

SOLUTION ✓ Using the chain rule for exponential functions, we compute the derivative Q' of the function Q. Thus,

$$\begin{aligned} Q'(t) &= Q_0e^{kt}\frac{d}{dt}(kt) \\ &= Q_0e^{kt}(k) \\ &= kQ_0e^{kt} \\ &= kQ(t) \qquad (Q(t) = Q_0e^{kt}) \end{aligned}$$

which is the desired conclusion. ■■■■

EXAMPLE 6 Find the points of inflection of the function $f(x) = e^{-x^2}$.

SOLUTION ✓ The first derivative of f is

$$f'(x) = -2xe^{-x^2}$$

Differentiating $f'(x)$ with respect to x yields

$$\begin{aligned} f''(x) &= (-2x)(-2xe^{-x^2}) - 2e^{-x^2} \\ &= 2e^{-x^2}(2x^2 - 1) \end{aligned}$$

Setting $f''(x) = 0$ gives

$$2e^{-x^2}(2x^2 - 1) = 0$$

Since e^{-x^2} never equals zero for any real value of x, we see that $x = \pm 1/\sqrt{2}$ are the only candidates for inflection points of f. The sign diagram of f'', shown in Figure 13.11, tells us that both $x = -1/\sqrt{2}$ and $x = 1/\sqrt{2}$ give rise to inflection points of f.

Next,

$$f\left(-\frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}\right) = e^{-1/2}$$

and the inflection points of f are $(-1/\sqrt{2}, e^{-1/2})$ and $(1/\sqrt{2}, e^{-1/2})$. The graph of f appears in Figure 13.12.

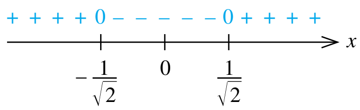


FIGURE 13.11
Sign diagram for f''

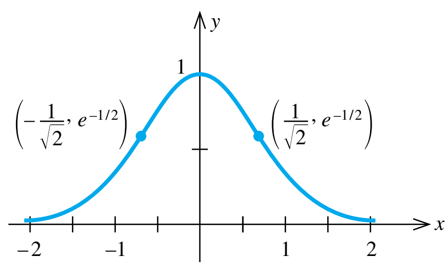


FIGURE 13.12
The graph of $y = e^{-x^2}$ has two inflection points.

APPLICATION

Our final example involves finding the absolute maximum of an exponential function.

EXAMPLE 7 Refer to Example 8, Section 13.1. The present value of the market price of the Blakely Office Building is given by

$$P(t) = 300{,}000e^{-0.09t+\sqrt{t}/2} \qquad (0 \le t \le 10)$$

Find the optimal present value of the building's market price.

SOLUTION ✔ To find the maximum value of P over [0, 10], we compute

$$P'(t) = 300{,}000e^{-0.09t+\sqrt{t}/2}\frac{d}{dt}\left(-0.09t + \frac{1}{2}t^{1/2}\right)$$

$$= 300{,}000e^{-0.09t+\sqrt{t}/2}\left(-0.09 + \frac{1}{4}t^{-1/2}\right)$$

Setting $P'(t) = 0$ gives

$$-0.09 + \frac{1}{4t^{1/2}} = 0$$

since $e^{-0.09t+\sqrt{t}/2}$ is never zero for any value of t. Solving this equation, we find

$$\frac{1}{4t^{1/2}} = 0.09$$

$$t^{1/2} = \frac{1}{4(0.09)}$$

$$= \frac{1}{0.36}$$

$$t \approx 7.72$$

the sole critical point of the function P. Finally, evaluating $P(t)$ at the critical point as well as at the end points of [0, 10], we have

t	$P(t)$
0	300,000
7.72	600,779
10	592,838

We conclude, accordingly, that the optimal present value of the property's market price is $600,779 and that this will occur 7.72 years from now.

Self-Check Exercises 13.3

1. Let $f(x) = xe^{-x}$.
  a. Find the first and second derivatives of f.
  b. Find the relative extrema of f.
  c. Find the inflection points of f.

2. An industrial asset is being depreciated at a rate so that its book value t yr from now will be

$$V(t) = 50{,}000e^{-0.4t}$$

dollars. How fast will the book value of the asset be changing 3 yr from now?

Solutions to Self-Check Exercises 13.3 can be found on page 918.

13.3 Exercises

In Exercises 1–28, find the derivative of the function.

1. $f(x) = e^{3x}$

2. $f(x) = 3e^{x}$

3. $g(t) = e^{-t}$

4. $f(x) = e^{-2x}$

5. $f(x) = e^{x} + x$

6. $f(x) = 2e^{x} - x^2$

7. $f(x) = x^3e^{x}$

8. $f(u) = u^2e^{-u}$

9. $f(x) = \dfrac{2e^{x}}{x}$

10. $f(x) = \dfrac{x}{e^{x}}$

11. $f(x) = 3(e^{x} + e^{-x})$

12. $f(x) = \dfrac{e^{x} + e^{-x}}{2}$

13. $f(w) = \dfrac{e^w + 1}{e^w}$ **14.** $f(x) = \dfrac{e^x}{e^x + 1}$

15. $f(x) = 2e^{3x-1}$ **16.** $f(t) = 4e^{3t+2}$

17. $h(x) = e^{-x^2}$ **18.** $f(x) = e^{x^2-1}$

19. $f(x) = 3e^{-1/x}$ **20.** $f(x) = e^{1/(2x)}$

21. $f(x) = (e^x + 1)^{25}$ **22.** $f(x) = (4 - e^{-3x})^3$

23. $f(x) = e^{\sqrt{x}}$ **24.** $f(t) = -e^{-\sqrt{2t}}$

25. $f(x) = (x - 1)e^{3x+2}$ **26.** $f(s) = (s^2 + 1)e^{-s^2}$

27. $f(x) = \dfrac{e^x - 1}{e^x + 1}$ **28.** $g(t) = \dfrac{e^{-t}}{1 + t^2}$

In Exercises 29–32, find the second derivative of the function.

29. $f(x) = e^{-4x} + 2e^{3x}$ **30.** $f(t) = 3e^{-2t} - 5e^{-t}$

31. $f(x) = 2xe^{3x}$ **32.** $f(t) = t^2e^{-2t}$

33. Find an equation of the tangent line to the graph of $y = e^{2x-3}$ at the point $(\frac{3}{2}, 1)$.

34. Find an equation of the tangent line to the graph of $y = e^{-x^2}$ at the point $(1, 1/e)$.

35. Determine the intervals where the function $f(x) = e^{-x^2/2}$ is increasing and where it is decreasing.

36. Determine the intervals where the function $f(x) = x^2e^{-x}$ is increasing and where it is decreasing.

37. Determine the intervals of concavity for the function $f(x) = \dfrac{e^x - e^{-x}}{2}$.

38. Determine the intervals of concavity for the function $f(x) = xe^x$.

39. Find the inflection point of the function $f(x) = xe^{-2x}$.

40. Find the inflection point(s) of the function $f(x) = 2e^{-x^2}$.

In Exercises 41–44, find the absolute extrema of the function.

41. $f(x) = e^{-x^2}$ on $[-1, 1]$

42. $h(x) = e^{x^2-4}$ on $[-2, 2]$

43. $g(x) = (2x - 1)e^{-x}$ on $[0, \infty)$

44. $f(x) = xe^{-x^2}$ on $[0, 2]$

In Exercises 45–48, use the curve-sketching guidelines of Chapter 12, page 837, to sketch the graph of the function.

45. $f(t) = e^t - t$ **46.** $h(x) = \dfrac{e^x + e^{-x}}{2}$

47. $f(x) = 2 - e^{-x}$ **48.** $f(x) = \dfrac{3}{1 + e^{-x}}$

A calculator is recommended for Exercises 49–59.

49. Sales Promotion The Lady Bug, a women's clothing chain store, found that t days after the end of a sales promotion the volume of sales was given by

$$S(t) = 20{,}000(1 + e^{-0.5t}) \qquad (0 \le t \le 5)$$

dollars. Find the rate of change of The Lady Bug's sales volume when $t = 1$, $t = 2$, $t = 3$, and $t = 4$.

50. Energy Consumption of Appliances The average energy consumption of the typical refrigerator/freezer manufactured by York Industries is approximately

$$C(t) = 1486e^{-0.073t} + 500 \qquad (0 \le t \le 20)$$

kilowatt-hours (kWh) per year, where t is measured in years, with $t = 0$ corresponding to 1972.

a. What was the average energy consumption of the York refrigerator/freezer at the beginning of 1972?

b. Prove that the average energy consumption of the York refrigerator/freezer is decreasing over the years in question.

c. All refrigerator/freezers manufactured as of January 1, 1990, must meet the 950-kWh/year maximum energy-consumption standard set by the National Appliance Conservation Act. Show that the York refrigerator/freezer satisfies this requirement.

51. Polio Immunization Polio, a once-feared killer, declined markedly in the United States in the 1950s after Jonas Salk developed the inactivated polio vaccine and mass immunization of children took place. The number of polio cases in the United States from the beginning of 1959 to the beginning of 1963 is approximated by the function

$$N(t) = 5.3e^{0.095t^2 - 0.85t} \qquad (0 \le t \le 4)$$

where $N(t)$ gives the number of polio cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1959.

a. Show that the function N is decreasing over the time interval under consideration.

b. How fast was the number of polio cases decreasing at the beginning of 1959? At the beginning of 1962? (*Comment:* Following the introduction of the oral vaccine developed by Dr. Albert B. Sabin in 1963, polio in the United States has, for all practical purposes, been eliminated.)

52. Blood Alcohol Level The percentage of alcohol in a person's bloodstream t hr after drinking 8 fluid oz of whiskey is given by

$$A(t) = 0.23te^{-0.4t} \qquad (0 \leq t \leq 12)$$

a. What is the percentage of alcohol in a person's bloodstream after $\frac{1}{2}$ hr? After 8 hr?
b. How fast is the percentage of alcohol in a person's bloodstream changing after $\frac{1}{2}$ hr? After 8 hr?
Source: Encyclopedia Britannica

53. Price of Perfume The monthly demand for a certain brand of perfume is given by the demand equation

$$p = 100e^{-0.0002x} + 150$$

where p denotes the retail unit price (in dollars) and x denotes the quantity (in 1-oz bottles) demanded.
a. Find the rate of change of the price per bottle when $x = 1000$ and when $x = 2000$.
b. What is the price per bottle when $x = 1000$? When $x = 2000$?

54. Price of Wine The monthly demand for a certain brand of table wine is given by the demand equation

$$p = 240\left(1 - \frac{3}{3 + e^{-0.0005x}}\right)$$

where p denotes the wholesale price per case (in dollars) and x denotes the number of cases demanded.
a. Find the rate of change of the price per case when $x = 1000$.
b. What is the price per case when $x = 1000$?

55. Spread of an Epidemic During a flu epidemic, the total number of students on a state university campus who had contracted influenza by the xth day was given by

$$N(x) = \frac{3000}{1 + 99e^{-x}} \qquad (x \geq 0)$$

a. How many students had influenza initially?
b. Derive an expression for the rate at which the disease was being spread and prove that the function N is increasing on the interval $(0, \infty)$.
c. Sketch the graph of N. What was the total number of students who contracted influenza during that particular epidemic?

56. Maximum Oil Production It has been estimated that the total production of oil from a certain oil well is given by

$$T(t) = -1000(t + 10)e^{-0.1t} + 10{,}000$$

thousand barrels t years after production has begun. Determine the year when the oil well will be producing at maximum capacity.

57. Optimal Selling Time Refer to Exercise 36, page 893. The present value of a piece of waterfront property purchased by an investor is given by the function

$$P(t) = 80{,}000e^{\sqrt{t}/2-0.09t} \qquad (0 \leq t \leq 8)$$

Determine the optimal time (based on present value) for the investor to sell the property. What is the property's optimal present value?

58. Oil Used to Fuel Productivity A study on worldwide oil use was prepared for a major oil company. The study predicted that the amount of oil used to fuel productivity in a certain country is given by

$$f(t) = 1.5 + 1.8te^{-1.2t} \qquad (0 \leq t \leq 4)$$

where $f(t)$ denotes the number of barrels per \$1000 of economic output and t is measured in decades ($t = 0$ corresponds to 1965). Compute $f'(0)$, $f'(1)$, $f'(2)$, and $f'(3)$ and interpret your results.

59. Percentage of Population Relocating Based on data obtained from the Census Bureau, the manager of Plymouth Van Lines estimates that the percentage of the total population relocating in year t ($t = 0$ corresponds to the year 1960) may be approximated by the formula

$$P(t) = 20.6e^{-0.009t} \qquad (0 \leq t \leq 35)$$

Compute $P'(10)$, $P'(20)$, and $P'(30)$ and interpret your results.

60. Price of a Commodity The price of a certain commodity in dollars per unit at time t (measured in weeks) is given by $p = 18 - 3e^{-2t} - 6e^{-t/3}$.
a. What is the price of the commodity at $t = 0$?
b. How fast is the price of the commodity changing at $t = 0$?
c. Find the equilibrium price of the commodity.
Hint: It is given by $\lim_{t\to\infty} p$.

61. Price of a Commodity The price of a certain commodity in dollars per unit at time t (measured in weeks) is given by $p = 8 + 4e^{-2t} + te^{-2t}$.
a. What is the price of the commodity at $t = 0$?
b. How fast is the price of the commodity changing at $t = 0$?
c. Find the equilibrium price of the commodity.
Hint: It's given by $\lim_{t\to\infty} p$. Also, use the fact that $\lim_{t\to\infty} te^{-2t} = 0$.

(continued on p. 918)

Using Technology

EXAMPLE 1 At the beginning of Section 13.3, we demonstrated via a table of values of $(e^h - 1)/h$ for selected values of h the plausibility of the result

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

To obtain a visual confirmation of this result, we plot the graph of

$$f(x) = \frac{e^x - 1}{x}$$

in the viewing rectangle $[-1, 1] \times [0, 2]$ (Figure T1). From the graph of f, we see that $f(x)$ appears to approach 1 as x approaches 0.

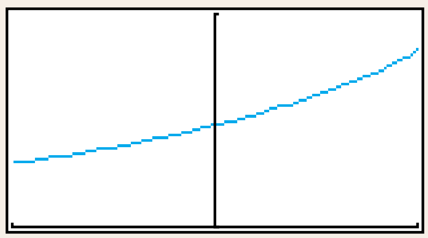


FIGURE T1
The graph of f in the viewing rectangle $[-1, 1] \times [0, 2]$

The numerical derivative function of a graphing utility will yield the derivative of an exponential or logarithmic function for any value of x, just as it did for algebraic functions.*

* The rules for differentiating logarithmic functions will be covered in Section 13.4. However, the exercises given here can be done without using these rules.

Exercises

In Exercises 1–6, use the numerical derivative operation of a graphing utility to find the rate of change of $f(x)$ at the given value of x. Give your answer accurate to four decimal places.

1. $f(x) = x^3 e^{-1/x}$; $x = -1$
2. $f(x) = (\sqrt{x} + 1)^{3/2} e^{-x}$; $x = 0.5$
3. $f(x) = x^3 \sqrt{\ln x}$; $x = 2$
4. $f(x) = \dfrac{\sqrt{x} \ln x}{x + 1}$; $x = 3.2$
5. $f(x) = e^{-x} \ln(2x + 1)$; $x = 0.5$
6. $f(x) = \dfrac{e^{-\sqrt{x}}}{\ln(x^2 + 1)}$; $x = 1$

7. **An Extinction Situation** The number of saltwater crocodiles in a certain area of northern Australia is given by

$$P(t) = \frac{300e^{-0.024t}}{5e^{-0.024t} + 1}$$

a. How many crocodiles were in the population initially?
b. Show that $\lim_{t \to \infty} P(t) = 0$.
c. Use a graphing calculator to plot the graph of P in the viewing rectangle $[0, 200] \times [0, 70]$.
(*Comment:* This phenomenon is referred to as an *extinction situation.*)

8. **Income of American Families** Based on data compiled by the House Budget Committee, the House Ways and Means Committee, and the U.S. Census Bureau, it is

estimated that the number of American families y (in millions) who earned x thousand dollars in 1990 is related by the equation

$$y = 0.1584xe^{-0.0000016x^3+0.00011x^2-0.04491x} \qquad (x > 0)$$

a. Use a graphing utility to plot the graph of the equation in the viewing rectangle $[0, 150] \times [0, 2]$.
b. How fast is y changing with respect to x when $x = 10$? When $x = 50$? Interpret your results.
Source: House Budget Committee, House Ways and Means Committee, and U.S. Census Bureau

9. **World Population Growth** According to a study conducted by the United Nations Population Division, the world population (in billions) is approximated by the function

$$f(t) = \frac{12}{1 + 3.74914e^{-1.42804t}} \qquad (0 \leq t \leq 4)$$

where t is measured in half-centuries, with $t = 0$ corresponding to the beginning of 1950.
a. Use a graphing utility to plot the graph of f in the viewing rectangle $[0, 5] \times [0, 14]$.
b. How fast was the world population expected to increase at the beginning of the year 2000?
Source: United Nations Population Division

10. **Loan Amortization** The Sotos plan to secure a loan of \$160,000 to purchase a house. They are considering a conventional 30-yr home mortgage at 9%/year on the unpaid balance. It can be shown that the Sotos will have an outstanding principal of

$$B(x) = \frac{160{,}000(1.0075^{360} - 1.0075^{x})}{1.0075^{360} - 1}$$

dollars after making x monthly payments of \$1287.40.
a. Use a graphing utility to plot the graph of $B(x)$, using the viewing rectangle $[0, 360] \times [0, 160{,}000]$.
b. Compute $B(0)$ and $B'(0)$ and interpret your results; compute $B(180)$ and $B'(180)$ and interpret your results.

11. **Increase in Juvenile Offenders** The number of youths aged 15 to 19 will increase by 21% between 1994 and 2005, pushing up the crime rate. According to the National Council on Crime and Delinquency, the number of violent crime arrests of juveniles under age 18 in year t is given by

$$f(t) = -0.438t^2 + 9.002t + 107 \qquad (0 \leq t \leq 13)$$

where $f(t)$ is measured in thousands and t in years, with $t = 0$ corresponding to 1989. According to the same source, if trends like inner-city drug use and wider availability of guns continues, then the number of violent crime arrests of juveniles under age 18 in year t will be given by

$$g(t) = \begin{cases} -0.438t^2 + 9.002t + 107 & \text{if } 0 \leq t < 4 \\ 99.456e^{0.07824t} & \text{if } 4 \leq t \leq 13 \end{cases}$$

where $g(t)$ is measured in thousands and $t = 0$ corresponds to 1989.
a. Compute $f(11)$ and $g(11)$ and interpret your results.
b. Compute $f'(11)$ and $g'(11)$ and interpret your results.
Source: National Council on Crime and Delinquency

12. **Increasing Crop Yields** If left untreated on bean stems, aphids (small insects that suck plant juices) will multiply at an increasing rate during the summer months and reduce productivity and crop yield of cultivated crops. But if the aphids are treated in mid-June, the numbers decrease sharply to less than 100/bean stem, allowing for steep rises in crop yield. The function

$$F(t) = \begin{cases} 62e^{1.152t} & \text{if } 0 \leq t < 1.5 \\ 349e^{-1.324(t-1.5)} & \text{if } 1.5 \leq t \leq 3 \end{cases}$$

gives the number of aphids in a typical bean stem at time t, where t is measured in months, with $t = 0$ corresponding to the beginning of May.
a. How many aphids are there on a typical bean stem at the beginning of June ($t = 1$)? At the beginning of July ($t = 2$)?
b. How fast is the population of aphids changing at the beginning of June? At the beginning of July?
Source: The Random House Encyclopedia

13. **Percentage of Females in the Labor Force** Based on data from the U.S. Census Bureau, the chief economist of Manpower, Inc., constructed the following formula giving the percentage of the total female population in the civilian labor force, $P(t)$, at the beginning of the tth decade ($t = 0$ corresponds to the year 1900):

$$P(t) = \frac{74}{1 + 2.6e^{-0.166t+0.04536t^2-0.0066t^3}} \qquad (0 \leq t \leq 11)$$

Assume this trend continued for the rest of the twentieth century.
a. What was the percentage of the total female population in the civilian labor force at the beginning of the year 2000?
b. What was the growth rate of the percentage of the total female population in the civilian labor force at the beginning of the year 2000?
Source: U.S. Census Bureau

62. **Absorption of Drugs** A liquid carries a drug into an organ of volume V cm^3 at the rate of a cm^3/sec and leaves at the same rate. The concentration of the drug in the entering liquid is c g/cm^3. Letting $x(t)$ denote the concentration of the drug in the organ at any time t, we have $x(t) = c(1 - e^{-at/V})$.
   a. Show that x is an increasing function on $(0, \infty)$.
   b. Sketch the graph of x.

63. **Absorption of Drugs** Refer to Exercise 62. Suppose the maximum concentration of the drug in the organ must *not* exceed m g/cm^3, where $m < c$. Show that the liquid must not be allowed to enter the organ for a time longer than

$$T = \left(\frac{V}{a}\right) \ln \left(\frac{c}{c - m}\right)$$

minutes.

In Exercises 64–67, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

64. If $f(x) = 3^x$, then $f'(x) = x \cdot 3^{x-1}$.

65. If $f(x) = e^{\pi}$, then $f'(x) = e^{\pi}$.

66. If $f(x) = \pi^x$, then $f'(x) = \pi^x$.

67. If $x^2 + e^y = 10$, then $y' = \dfrac{-2x}{e^y}$.

Solutions to Self-Check Exercises 13.3

1. **a.** Using the product rule, we obtain

$$f'(x) = x\frac{d}{dx}e^{-x} + e^{-x}\frac{d}{dx}x$$
$$= -xe^{-x} + e^{-x} = (1 - x)e^{-x}$$

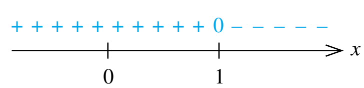
Using the product rule once again, we obtain

$$f''(x) = (1 - x)\frac{d}{dx}e^{-x} + e^{-x}\frac{d}{dx}(1 - x)$$
$$= (1 - x)(-e^{-x}) + e^{-x}(-1)$$
$$= -e^{-x} + xe^{-x} - e^{-x} = (x - 2)e^{-x}$$

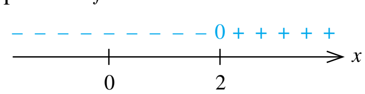
b. Setting $f'(x) = 0$ gives

$$(1 - x)e^{-x} = 0$$

Since $e^{-x} \neq 0$, we see that $1 - x = 0$, and this gives $x = 1$ as the only critical point of f. The sign diagram of f' shown in the accompanying figure tells us that the point $(1, e^{-1})$ is a relative maximum of f.



c. Setting $f''(x) = 0$ gives $x - 2 = 0$, so $x = 2$ is a candidate for an inflection point of f. The sign diagram of f'' (accompanying figure) shows that $(2, 2e^{-2})$ is an inflection point of f.



2. The rate of change of the book value of the asset t yr from now is

$$V'(t) = 50{,}000 \frac{d}{dt} e^{-0.4t}$$
$$= 50{,}000(-0.4)e^{-0.4t} = -20{,}000e^{-0.4t}$$

Therefore, 3 yr from now the book value of the asset will be changing at the rate of

$$V'(3) = -20{,}000e^{-0.4(3)} = -20{,}000e^{-1.2} \approx -6023.88$$

—that is, decreasing at the rate of approximately \$6024/year.

13.4 Differentiation of Logarithmic Functions

The Derivative of ln x

Let's now turn our attention to the differentiation of logarithmic functions.

Rule 3: Derivative of ln x

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \qquad (x \neq 0)$$

To derive Rule 3, suppose $x > 0$ and write $f(x) = \ln x$ in the equivalent form

$$x = e^{f(x)}$$

Differentiating both sides of the equation with respect to x, we find, using the chain rule,

$$1 = e^{f(x)} \cdot f'(x)$$

from which we see that

$$f'(x) = \frac{1}{e^{f(x)}}$$

or, since $e^{f(x)} = x$,

$$f'(x) = \frac{1}{x}$$

as we set out to show. You are asked to prove the rule for the case $x < 0$ in Exercise 61.

EXAMPLE 1 Compute the derivative of each of the following functions:

a. $f(x) = x \ln x$ **b.** $g(x) = \dfrac{\ln x}{x}$

SOLUTION ✓

a. Using the product rule, we obtain

$$f'(x) = \frac{d}{dx}(x \ln x) = x \frac{d}{dx}(\ln x) + (\ln x)\frac{d}{dx}(x)$$
$$= x\left(\frac{1}{x}\right) + \ln x = 1 + \ln x$$

b. Using the quotient rule, we obtain

$$g'(x) = \frac{x \dfrac{d}{dx}(\ln x) - (\ln x)\dfrac{d}{dx}(x)}{x^2} = \frac{x\left(\dfrac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Group Discussion

You can derive the formula for the derivative of $f(x) = \ln x$ directly from the definition of the derivative, as follows.

1. Show that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \ln\left(1 + \frac{h}{x}\right)^{1/h}$$

2. Put $m = x/h$ and note that $m \to \infty$ as $h \to 0$. Furthermore, $f'(x)$ can be written in the form

$$f'(x) = \lim_{m \to \infty} \ln\left(1 + \frac{1}{m}\right)^{m/x}$$

3. Finally, use both the fact that the natural logarithmic function is continuous and the definition of the number e to show that

$$f'(x) = \frac{1}{x} \ln\left[\lim_{m \to \infty}\left(1 + \frac{1}{m}\right)^m\right] = \frac{1}{x}$$

The Chain Rule for Logarithmic Functions

To enlarge the class of logarithmic functions to be differentiated, we appeal once more to the chain rule to obtain the following rule for differentiating composite functions of the form $h(x) = \ln f(x)$, where $f(x)$ is assumed to be a positive differentiable function.

Rule 4: Chain Rule for Logarithmic Functions

If $f(x)$ is a differentiable function, then

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)} \qquad [f(x) > 0]$$

To see this, observe that $h(x) = g[f(x)]$, where $g(x) = \ln x$ $(x > 0)$. Since $g'(x) = 1/x$, we have, using the chain rule,

$$\begin{aligned} h'(x) &= g'(f(x))f'(x) \\ &= \frac{1}{f(x)} f'(x) = \frac{f'(x)}{f(x)} \end{aligned}$$

Observe that in the special case $f(x) = x$, $h(x) = \ln x$, so the derivative of h is, by Rule 3, given by $h'(x) = 1/x$.

EXAMPLE 2 Find the derivative of the function $f(x) = \ln(x^2 + 1)$.

SOLUTION ✔ Using Rule 4, we see immediately that

$$f'(x) = \frac{\frac{d}{dx}(x^2+1)}{x^2+1} = \frac{2x}{x^2+1}$$

■■■■

When differentiating functions involving logarithms, the rules of logarithms may be used to advantage, as shown in Examples 3 and 4.

EXAMPLE 3 Differentiate the function $y = \ln[(x^2 + 1)(x^3 + 2)^6]$.

SOLUTION ✔ We first rewrite the given function using the properties of logarithms:

$$\begin{aligned} y &= \ln[(x^2+1)(x^3+2)^6] \\ &= \ln(x^2+1) + \ln(x^3+2)^6 && (\ln mn = \ln m + \ln n) \\ &= \ln(x^2+1) + 6\ln(x^3+2) && (\ln m^n = n \ln m) \end{aligned}$$

Differentiating and using Rule 4, we obtain

$$\begin{aligned} y' &= \frac{\frac{d}{dx}(x^2+1)}{x^2+1} + \frac{6\frac{d}{dx}(x^3+2)}{x^3+2} \\ &= \frac{2x}{x^2+1} + \frac{6(3x^2)}{x^3+2} = \frac{2x}{x^2+1} + \frac{18x^2}{x^3+2} \end{aligned}$$

■■■■

Exploring with Technology

Use a graphing utility to plot the graphs of $f(x) = \ln x$; its first derivative function, $f'(x) = 1/x$; and its second derivative function, $f''(x) = -1/x^2$, using the same viewing rectangle $[0, 4] \times [-3, 3]$.

1. Describe the properties of the graph of f revealed by studying the graph of $f'(x)$. What can you say about the rate of increase of f for large values of x?
2. Describe the properties of the graph of f revealed by studying the graph of $f''(x)$. What can you say about the concavity of f for large values of x?

EXAMPLE 4 Find the derivative of the function $g(t) = \ln(t^2 e^{-t^2})$.

SOLUTION ✔ Here again, to save a lot of work, we first simplify the given expression using the properties of logarithms. We have

$$\begin{aligned} g(t) &= \ln(t^2 e^{-t^2}) \\ &= \ln t^2 + \ln e^{-t^2} && (\ln mn = \ln m + \ln n) \\ &= 2 \ln t - t^2 && (\ln m^n = n \ln m \text{ and } \ln e = 1) \end{aligned}$$

Therefore,

$$g'(t) = \frac{2}{t} - 2t = \frac{2(1 - t^2)}{t}$$

■■■■

Logarithmic Differentiation

As we saw in the last two examples, the task of finding the derivative of a given function can be made easier by first applying the laws of logarithms to simplify the function. We now illustrate a process called **logarithmic differentiation,** which not only simplifies the calculation of the derivatives of certain functions but also enables us to compute the derivatives of functions we could not otherwise differentiate using the techniques developed thus far.

EXAMPLE 5 Differentiate $y = x(x + 1)(x^2 + 1)$, using logarithmic differentiation.

SOLUTION ✔ First, we take the natural logarithm on both sides of the given equation, obtaining

$$\ln y = \ln x(x + 1)(x^2 + 1)$$

Next, we use the properties of logarithms to rewrite the right-hand side of this equation, obtaining

$$\ln y = \ln x + \ln(x + 1) + \ln(x^2 + 1)$$

If we differentiate both sides of this equation, we have

$$\begin{aligned} \frac{d}{dx} \ln y &= \frac{d}{dx} [\ln x + \ln(x + 1) + \ln(x^2 + 1)] \\ &= \frac{1}{x} + \frac{1}{x + 1} + \frac{2x}{x^2 + 1} && \text{(Using Rule 4)} \end{aligned}$$

To evaluate the expression on the left-hand side, note that y is a function of x. Therefore, writing $y = f(x)$ to remind us of this fact, we have

$$\begin{aligned}\frac{d}{dx}\ln y &= \frac{d}{dx}\ln[f(x)] && [\text{Writing } y = f(x)]\\ &= \frac{f'(x)}{f(x)} && (\text{Using Rule 4})\\ &= \frac{y'}{y} && [\text{Returning to using } y \text{ instead of } f(x)]\end{aligned}$$

Therefore, we have

$$\frac{y'}{y} = \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}$$

Finally, solving for y', we have

$$\begin{aligned}y' &= y\left(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}\right)\\ &= x(x+1)(x^2+1)\left(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}\right)\end{aligned}$$

■■■■

Before considering other examples, let's summarize the important steps involved in logarithmic differentiation.

Finding $\frac{dy}{dx}$ by Logarithmic Differentiation

1. Take the natural logarithm on both sides of the equation and use the properties of logarithms to write any "complicated expression" as a sum of simpler terms.
2. Differentiate both sides of the equation with respect to x.
3. Solve the resulting equation for $\dfrac{dy}{dx}$.

EXAMPLE 6 Differentiate $y = x^2(x-1)(x^2+4)^3$.

SOLUTION ✓ Taking the natural logarithm on both sides of the given equation and using the laws of logarithms, we obtain

$$\begin{aligned}\ln y &= \ln x^2(x-1)(x^2+4)^3\\ &= \ln x^2 + \ln(x-1) + \ln(x^2+4)^3\\ &= 2\ln x + \ln(x-1) + 3\ln(x^2+4)\end{aligned}$$

Differentiating both sides of the equation with respect to x, we have

$$\frac{d}{dx}\ln y = \frac{y'}{y} = \frac{2}{x} + \frac{1}{x-1} + 3\cdot\frac{2x}{x^2+4}$$

Finally, solving for y', we have

$$y' = y\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}\right)$$
$$= x^2(x-1)(x^2+4)^3\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}\right)$$

EXAMPLE 7 Find the derivative of $f(x) = x^x (x > 0)$.

SOLUTION ✔ A word of caution! This function is neither a power function nor an exponential function. Taking the natural logarithm on both sides of the equation gives

$$\ln f(x) = \ln x^x = x \ln x$$

Differentiating both sides of the equation with respect to x, we obtain

$$\frac{f'(x)}{f(x)} = x\frac{d}{dx}\ln x + (\ln x)\frac{d}{dx}x$$
$$= x\left(\frac{1}{x}\right) + \ln x$$
$$= 1 + \ln x$$

Therefore,

$$f'(x) = f(x)(1 + \ln x) = x^x(1 + \ln x)$$

Exploring with Technology

Refer to Example 7.

1. Use a graphing utility to plot the graph of $f(x) = x^x$, using the viewing rectangle $[0, 2] \times [0, 2]$. Then use **ZOOM** and **TRACE** to show that

$$\lim_{x\to 0^+} f(x) = 1$$

2. Use the results of part 1 and Example 7 to show that $\lim_{x\to 0^+} f'(x) = -\infty$. Justify your answer.

SELF-CHECK EXERCISES 13.4

1. Find an equation of the tangent line to the graph of $f(x) = x \ln(2x + 3)$ at the point $(-1, 0)$.
2. Use logarithmic differentiation to compute y', given $y = (2x + 1)^3(3x + 4)^5$.

Solutions to Self-Check Exercises 13.4 can be found on page 926.

13.4 Exercises

In Exercises 1–32, find the derivative of the function.

1. $f(x) = 5 \ln x$
2. $f(x) = \ln 5x$
3. $f(x) = \ln(x + 1)$
4. $g(x) = \ln(2x + 1)$
5. $f(x) = \ln x^8$
6. $h(t) = 2 \ln t^5$
7. $f(x) = \ln \sqrt{x}$
8. $f(x) = \ln(\sqrt{x} + 1)$
9. $f(x) = \ln \dfrac{1}{x^2}$
10. $f(x) = \ln \dfrac{1}{2x^3}$
11. $f(x) = \ln(4x^2 - 6x + 3)$
12. $f(x) = \ln(3x^2 - 2x + 1)$
13. $f(x) = \ln \dfrac{2x}{x + 1}$
14. $f(x) = \ln \dfrac{x + 1}{x - 1}$
15. $f(x) = x^2 \ln x$
16. $f(x) = 3x^2 \ln 2x$
17. $f(x) = \dfrac{2 \ln x}{x}$
18. $f(x) = \dfrac{3 \ln x}{x^2}$
19. $f(u) = \ln(u - 2)^3$
20. $f(x) = \ln(x^3 - 3)^4$
21. $f(x) = \sqrt{\ln x}$
22. $f(x) = \sqrt{\ln x + x}$
23. $f(x) = (\ln x)^3$
24. $f(x) = 2(\ln x)^{3/2}$
25. $f(x) = \ln(x^3 + 1)$
26. $f(x) = \ln \sqrt{x^2 - 4}$
27. $f(x) = e^x \ln x$
28. $f(x) = e^x \ln \sqrt{x + 3}$
29. $f(t) = e^{2t} \ln(t + 1)$
30. $g(t) = t^2 \ln(e^{2t} + 1)$
31. $f(x) = \dfrac{\ln x}{x}$
32. $g(t) = \dfrac{t}{\ln t}$

In Exercises 33–36, find the second derivative of the function.

33. $f(x) = \ln 2x$
34. $f(x) = \ln(x + 5)$
35. $f(x) = \ln(x^2 + 2)$
36. $f(x) = (\ln x)^2$

In Exercises 37–46, use logarithmic differentiation to find the derivative of the function.

37. $y = (x + 1)^2(x + 2)^3$
38. $y = (3x + 2)^4(5x - 1)^2$
39. $y = (x - 1)^2(x + 1)^3(x + 3)^4$
40. $y = \sqrt{3x + 5}(2x - 3)^4$
41. $y = \dfrac{(2x^2 - 1)^5}{\sqrt{x + 1}}$
42. $y = \dfrac{\sqrt{4 + 3x^2}}{\sqrt[3]{x^2 + 1}}$
43. $y = 3^x$
44. $y = x^{x+2}$
45. $y = (x^2 + 1)^x$
46. $y = x^{\ln x}$
47. Find an equation of the tangent line to the graph of $y = x \ln x$ at the point (1, 0).
48. Find an equation of the tangent line to the graph of $y = \ln x^2$ at the point (2, ln 4).
49. Determine the intervals where the function $f(x) = \ln x^2$ is increasing and where it is decreasing.
50. Determine the intervals where the function $f(x) = \dfrac{\ln x}{x}$ is increasing and where it is decreasing.
51. Determine the intervals of concavity for the function $f(x) = x^2 + \ln x^2$.
52. Determine the intervals of concavity for the function $f(x) = \dfrac{\ln x}{x}$.
53. Find the inflection points of the function $f(x) = \ln(x^2 + 1)$.
54. Find the inflection points of the function $f(x) = x^2 \ln x$.
55. Find the absolute extrema of the function $f(x) = x - \ln x$ on $[\frac{1}{2}, 3]$.
56. Find the absolute extrema of the function $g(x) = \dfrac{x}{\ln x}$ on $[2, \infty)$.

In Exercises 57 and 58, use the guidelines on page 893 to sketch the graph of the given function.

57. $f(x) = \ln(x - 1)$
58. $f(x) = 2x - \ln x$

In Exercises 59 and 60, determine whether the statement is true or false. If it is true, explain why it is true. If it is false, give an example to show why it is false.

59. If $f(x) = \ln 5$, then $f'(x) = 1/5$.
60. If $f(x) = \ln a^x$, then $f'(x) = \ln a$.
61. Prove that $\dfrac{d}{dx} \ln |x| = \dfrac{1}{x}$ $(x \neq 0)$ for the case $x < 0$.
62. Use the definition of the derivative to show that

$$\lim_{x \to 0} \frac{\ln(x + 1)}{x} = 1$$

SOLUTIONS TO SELF-CHECK EXERCISES 13.4

1. The slope of the tangent line to the graph of f at any point $(x, f(x))$ lying on the graph of f is given by $f'(x)$. Using the product rule, we find

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x\ln(2x+3)] \\ &= x\frac{d}{dx}\ln(2x+3) + \ln(2x+3)\cdot\frac{d}{dx}(x) \\ &= x\left(\frac{2}{2x+3}\right) + \ln(2x+3)\cdot 1 \\ &= \frac{2x}{2x+3} + \ln(2x+3) \end{aligned}$$

In particular, the slope of the tangent line to the graph of f at the point $(-1, 0)$ is

$$f'(-1) = \frac{-2}{-2+3} + \ln 1 = -2$$

Therefore, using the point-slope form of the equation of a line, we see that a required equation is

$$\begin{aligned} y - 0 &= -2(x+1) \\ y &= -2x - 2 \end{aligned}$$

2. Taking the logarithm on both sides of the equation gives

$$\begin{aligned} \ln y &= \ln(2x+1)^3(3x+4)^5 \\ &= \ln(2x+1)^3 + \ln(3x+4)^5 \\ &= 3\ln(2x+1) + 5\ln(3x+4) \end{aligned}$$

Differentiating both sides of the equation with respect to x, keeping in mind that y is a function of x, we obtain

$$\begin{aligned} \frac{d}{dx}(\ln y) = \frac{y'}{y} &= 3\cdot\frac{2}{2x+1} + 5\cdot\frac{3}{3x+4} \\ &= 3\left[\frac{2}{2x+1} + \frac{5}{3x+4}\right] \\ &= \left(\frac{6}{2x+1} + \frac{15}{3x+4}\right) \end{aligned}$$

and

$$y' = (2x+1)^3(3x+4)^5\cdot\left(\frac{6}{2x+1} + \frac{15}{3x+4}\right)$$

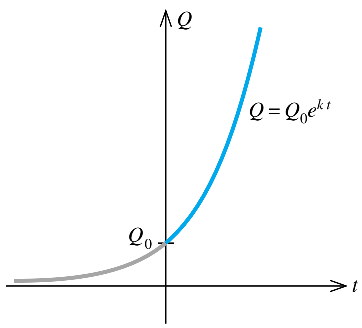
13.5 Exponential Functions as Mathematical Models

EXPONENTIAL GROWTH

Many problems arising from practical situations can be described mathematically in terms of exponential functions or functions closely related to the exponential function. In this section we look at some applications involving exponential functions from the fields of the life and social sciences.

In Section 13.1 we saw that the exponential function $f(x) = b^x$ is an increasing function when $b > 1$. In particular, the function $f(x) = e^x$ shares this property. From this result one may deduce that the function $Q(t) = Q_0e^{kt}$, where Q_0 and k are positive constants, has the following properties:

FIGURE 13.13
Exponential growth



1. $Q(0) = Q_0$
2. $Q(t)$ increases "rapidly" without bound as t increases without bound (Figure 13.13).

Property 1 follows from the computation

$$Q(0) = Q_0e^0 = Q_0$$

Next, to study the rate of change of the function $Q(t)$, we differentiate it with respect to t, obtaining

$$\begin{aligned} Q'(t) &= \frac{d}{dt}(Q_0e^{kt}) \\ &= Q_0\frac{d}{dt}(e^{kt}) \\ &= kQ_0e^{kt} \\ &= kQ(t) \end{aligned} \qquad \textbf{(7)}$$

Since $Q(t) > 0$ (because Q_0 is assumed to be positive) and $k > 0$, we see that $Q'(t) > 0$ and so $Q(t)$ is an increasing function of t. Our computation has in fact shed more light on an important property of the function $Q(t)$. Equation (7) says that the rate of increase of the function $Q(t)$ is proportional to the amount $Q(t)$ of the quantity present at time t. The implication is that as $Q(t)$ increases, so does the *rate of increase* of $Q(t)$, resulting in a very rapid increase in $Q(t)$ as t increases without bound.

Thus, the exponential function

$$Q(t) = Q_0e^{kt} \qquad (0 \le t < \infty) \qquad \textbf{(8)}$$

provides us with a mathematical model of a quantity $Q(t)$ that is initially present in the amount of $Q(0) = Q_0$ and whose rate of growth at any time t is directly proportional to the amount of the quantity present at time t. Such a quantity is said to exhibit **exponential growth,** and the constant k is called the **growth constant.** Interest earned on a fixed deposit when compounded continuously exhibits exponential growth. Other examples of exponential growth follow.

EXAMPLE 1 Under ideal laboratory conditions, the number of bacteria in a culture grows in accordance with the law $Q(t) = Q_0e^{kt}$, where Q_0 denotes the number of bacteria initially present in the culture, k is some constant determined by the strain of bacteria under consideration, and t is the elapsed time measured in hours. Suppose 10,000 bacteria are present initially in the culture and 60,000 present 2 hours later.

a. How many bacteria will there be in the culture at the end of 4 hours?
b. What is the rate of growth of the population after 4 hours?

SOLUTION ✔ **a.** We are given that $Q(0) = Q_0 = 10{,}000$, so $Q(t) = 10{,}000e^{kt}$. Next, the fact that 60,000 bacteria are present 2 hours later translates into $Q(2) = 60{,}000$. Thus,

$$60{,}000 = 10{,}000e^{2k}$$
$$e^{2k} = 6$$

Taking the natural logarithm on both sides of the equation, we obtain

$$\ln e^{2k} = \ln 6$$
$$2k = \ln 6 \qquad \text{(Since } \ln e = 1\text{)}$$
$$k \approx 0.8959$$

Thus, the number of bacteria present at any time t is given by

$$Q(t) = 10{,}000e^{0.8959t}$$

In particular, the number of bacteria present in the culture at the end of 4 hours is given by

$$Q(4) = 10{,}000e^{0.8959(4)}$$
$$= 360{,}029$$

b. The rate of growth of the bacteria population at any time t is given by

$$Q'(t) = kQ(t)$$

Thus, using the result from part (a), we find that the rate at which the population is growing at the end of 4 hours is

$$Q'(4) = kQ(4)$$
$$\approx (0.8959)(360{,}029)$$
$$\approx 322{,}550$$

or approximately 322,550 bacteria per hour. ■■■■

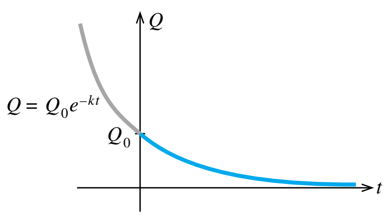
EXPONENTIAL DECAY

In contrast to exponential growth, a quantity exhibits **exponential decay** if it decreases at a rate that is directly proportional to its size. Such a quantity may be described by the exponential function

$$Q(t) = Q_0 e^{-kt} \qquad [t \in [0, \infty)] \tag{9}$$

where the positive constant Q_0 measures the amount present initially ($t = 0$) and k is some suitable positive number, called the **decay constant.** The choice of this number is determined by the nature of the substance under consideration. The graph of this function is sketched in Figure 13.14.

FIGURE 13.14 Exponential decay



To verify the properties ascribed to the function $Q(t)$, we simply compute

$$\begin{aligned} Q(0) &= Q_0 e^0 = Q_0 \\ Q'(t) &= \frac{d}{dt}(Q_0 e^{-kt}) \\ &= Q_0 \frac{d}{dt}(e^{-kt}) \\ &= -kQ_0 e^{-kt} = -kQ(t) \end{aligned}$$

EXAMPLE 2 Radioactive substances decay exponentially. For example, the amount of radium present at any time t obeys the law $Q(t) = Q_0 e^{-kt}$, where Q_0 is the initial amount present and k is a suitable positive constant. The **half-life of a radioactive substance** is the time required for a given amount to be reduced by one-half. Now, it is known that the half-life of radium is approximately 1600 years. Suppose initially there are 200 milligrams of pure radium. Find the amount left after t years. What is the amount left after 800 years?

SOLUTION ✔ The initial amount of radium present is 200 milligrams, so $Q(0) = Q_0 = 200$. Thus, $Q(t) = 200e^{-kt}$. Next, the datum concerning the half-life of radium implies that $Q(1600) = 100$, and this gives

$$\begin{aligned} 100 &= 200e^{-1600k} \\ e^{-1600k} &= \frac{1}{2} \end{aligned}$$

Taking the natural logarithm on both sides of this equation yields

$$\begin{aligned} -1600k \ln e &= \ln \frac{1}{2} \\ -1600k &= \ln \frac{1}{2} \qquad (\ln e = 1) \\ k &= -\frac{1}{1600} \ln\left(\frac{1}{2}\right) = 0.0004332 \end{aligned}$$

Therefore, the amount of radium left after t years is

$$Q(t) = 200e^{-0.0004332t}$$

In particular, the amount of radium left after 800 years is

$$Q(800) = 200e^{-0.0004332(800)} \approx 141.42$$

or approximately 141 milligrams. ■■■■

EXAMPLE 3 Carbon 14, a radioactive isotope of carbon, has a half-life of 5770 years. What is its decay constant?

SOLUTION ✔ We have $Q(t) = Q_0 e^{-kt}$. Since the half-life of the element is 5770 years, half of the substance is left at the end of that period. That is,

$$Q(5770) = Q_0 e^{-5770k} = \frac{1}{2} Q_0$$

$$e^{-5770k} = \frac{1}{2}$$

Taking the natural logarithm on both sides of this equation, we have

$$\ln e^{-5770k} = \ln \frac{1}{2}$$

$$-5770k = -0.693147$$

$$k \approx 0.00012$$

■■■■

Carbon-14 dating is a well-known method used by anthropologists to establish the age of animal and plant fossils. This method assumes that the proportion of carbon 14 (C-14) present in the atmosphere has remained constant over the past 50,000 years. Professor Willard Libby, recipient of the Nobel Prize in chemistry in 1960, proposed this theory.

The amount of C-14 in the tissues of a living plant or animal is constant. However, when an organism dies, it stops absorbing new quantities of C-14, and the amount of C-14 in the remains diminishes because of the natural decay of the radioactive substance. Thus, the approximate age of a plant or animal fossil can be determined by measuring the amount of C-14 present in the remains.

EXAMPLE 4 A skull from an archeological site has one-tenth the amount of C-14 that it originally contained. Determine the approximate age of the skull.

SOLUTION ✔ Here,

$$\begin{aligned} Q(t) &= Q_0 e^{-kt} \\ &= Q_0 e^{-0.00012t} \end{aligned}$$

where Q_0 is the amount of C-14 present originally and k, the decay constant, is equal to 0.00012 (see Example 3). Since $Q(t) = (1/10)Q_0$, we have

$$\frac{1}{10}Q_0 = Q_0e^{-0.00012t}$$

$$\ln\frac{1}{10} = -0.00012t \qquad \text{(Taking the natural logarithm on both sides)}$$

$$t = \frac{\ln\frac{1}{10}}{-0.00012} \approx 19{,}200$$

or approximately 19,200 years. ■■■■

Learning Curves

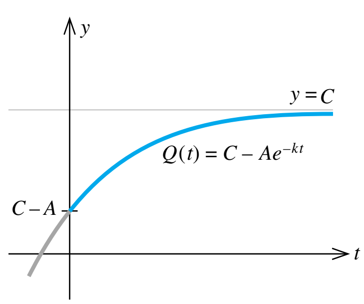
The next example shows how the exponential function may be applied to describe certain types of learning processes. Consider the function

$$Q(t) = C - Ae^{-kt}$$

where C, A, and k are positive constants. To sketch the graph of the function Q, observe that its y-intercept is given by $Q(0) = C - A$. Next, we compute

$$Q'(t) = kAe^{-kt}$$

FIGURE 13.15
A learning curve



Since both k and A are positive, we see that $Q'(t) > 0$ for all values of t. Thus, $Q(t)$ is an increasing function of t. Also,

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}(C - Ae^{-kt}) = \lim_{t\to\infty} C - \lim_{t\to\infty} Ae^{-kt} = C$$

so $y = C$ is a horizontal asymptote of Q. Thus, $Q(t)$ increases and approaches the number C as t increases without bound. The graph of the function Q is shown in Figure 13.15, where that part of the graph corresponding to the negative values of t is drawn with a gray line since, in practice, one normally restricts the domain of the function to the interval $[0, \infty)$.

Observe that $Q(t)$ $(t > 0)$ increases rather rapidly initially but that the rate of increase slows down considerably after a while. To see this, we compute

$$\lim_{t\to\infty} Q'(t) = \lim_{t\to\infty} kAe^{-kt} = 0$$

This behavior of the graph of the function Q closely resembles the learning pattern experienced by workers engaged in highly repetitive work. For example, the productivity of an assembly-line worker increases very rapidly in the early stages of the training period. This productivity increase is a direct result of the worker's training and accumulated experience. But the rate of increase of productivity slows as time goes by, and the worker's productivity level approaches some fixed level due to the limitations of the worker and the machine. Because of this characteristic, the graph of the function $Q(t) = C - Ae^{-kt}$ is often called a **learning curve.**

EXAMPLE 5 The Camera Division of the Eastman Optical Company produces a 35-mm single-lens reflex camera. Eastman's training department determines that after completing the basic training program, a new, previously inexperienced employee will be able to assemble

$$Q(t) = 50 - 30e^{-0.5t}$$

model F cameras per day, t months after the employee starts work on the assembly line.

a. How many model F cameras can a new employee assemble per day after basic training?
b. How many model F cameras can an employee with 1 month of experience assemble per day? An employee with 2 months of experience? An employee with 6 months of experience?
c. How many model F cameras can the average experienced employee assemble per day?

SOLUTION ✔

a. The number of model F cameras a new employee can assemble is given by

$$Q(0) = 50 - 30 = 20$$

b. The number of model F cameras that an employee with 1 month of experience, 2 months of experience, and 6 months of experience can assemble per day is given by

$$Q(1) = 50 - 30e^{-0.5} \approx 31.80$$

$$Q(2) = 50 - 30e^{-1} \approx 38.96$$

$$Q(6) = 50 - 30e^{-3} \approx 48.51$$

or approximately 32, 39, and 49, respectively.

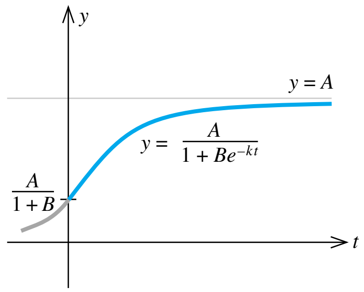
c. As t increases without bound, $Q(t)$ approaches 50. Hence, the average experienced employee can ultimately be expected to assemble 50 model F cameras per day.

Other applications of the learning curve are found in models that describe the dissemination of information about a product or the velocity of an object dropped into a viscous medium.

Logistic Growth Functions

Our last example of an application of exponential functions to the description of natural phenomena involves the **logistic** (also called the **S-shaped,** or **sigmoi-**

FIGURE 13.16
A logistic curve



dal) curve, which is the graph of the function

$$Q(t) = \frac{A}{1 + Be^{-kt}}$$

where A, B, and k are positive constants. The function Q is called a **logistic growth function,** and the graph of the function Q is sketched in Figure 13.16.

Observe that $Q(t)$ increases rather rapidly for small values of t. In fact, for small values of t, the logistic curve resembles an exponential growth curve. However, the *rate of growth* of $Q(t)$ decreases quite rapidly as t increases and $Q(t)$ approaches the number A as t increases without bound.

Thus, the logistic curve exhibits both the property of rapid growth of the exponential growth curve as well as the "saturation" property of the learning curve. Because of these characteristics, the logistic curve serves as a suitable mathematical model for describing many natural phenomena. For example, if a small number of rabbits were introduced to a tiny island in the South Pacific, the rabbit population might be expected to grow very rapidly at first, but the growth rate would decrease quickly as overcrowding, scarcity of food, and other environmental factors affected it. The population would eventually stabilize at a level compatible with the life-support capacity of the environment. Models describing the spread of rumors and epidemics are other examples of the application of the logistic curve.

EXAMPLE 6 The number of soldiers at Fort MacArthur who contracted influenza after t days during a flu epidemic is approximated by the exponential model

$$Q(t) = \frac{5000}{1 + 1249e^{-kt}}$$

If 40 soldiers contracted the flu by day 7, find how many soldiers contracted the flu by day 15.

SOLUTION ✓ The given information implies that

$$Q(7) = 40 \qquad \text{and} \qquad Q(7) = \frac{5000}{1 + 1249e^{-7k}} = 40$$

Thus,

$$40(1 + 1249e^{-7k}) = 5000$$

$$1 + 1249e^{-7k} = \frac{5000}{40} = 125$$

$$e^{-7k} = \frac{124}{1249}$$

$$-7k = \ln \frac{124}{1249}$$

$$k = -\frac{\ln \frac{124}{1249}}{7} \approx 0.33$$

Therefore, the number of soldiers who contracted the flu after t days is given by

$$Q(t) = \frac{5000}{1 + 1249e^{-0.33t}}$$

In particular, the number of soldiers who contracted the flu by day 15 is given by

$$Q(15) = \frac{5000}{1 + 1249e^{-15(0.33)}} \approx 508$$

or approximately 508 soldiers. ■■■■

Exploring with Technology

Refer to Example 6.

1. Use a graphing utility to plot the graph of the function Q, using the viewing rectangle $[0, 40] \times [0, 5000]$.
2. Find how long it takes for the first 1000 soldiers to contract the flu.

Hint: Plot the graphs of $y_1 = Q(t)$ and $y_2 = 1000$ and find the point of intersection of the two graphs.

SELF-CHECK EXERCISE 13.5

Suppose that the population (in millions) of a country at any time t grows in accordance with the rule

$$P = \left(P_0 + \frac{I}{k}\right)e^{kt} - \frac{I}{k}$$

where P denotes the population at any time t, k is a constant reflecting the natural growth rate of the population, I is a constant giving the (constant) rate of immigration into the country, and P_0 is the total population of the country at time $t = 0$. The population of the United States in the year 1980 ($t = 0$) was 226.5 million. If the natural growth rate is 0.8% annually ($k = 0.008$) and net immigration is allowed at the rate of half a million people per year ($I = 0.5$) until the end of the century, what is the population of the United States expected to be in the year 2005?

Solutions to Self-Check Exercise 13.5 can be found on page 938.

13.5 Exercises

A calculator is recommended for this exercise set.

1. **EXPONENTIAL GROWTH** Given that a quantity $Q(t)$ is described by the exponential growth function

$$Q(t) = 400e^{0.05t}$$

where t is measured in minutes, answer the following questions.
a. What is the growth constant?
b. What quantity is present initially?
c. Using a calculator, complete the following table of values:

t	0	10	20	100	1000
Q					

2. **EXPONENTIAL DECAY** Given that a quantity $Q(t)$ exhibiting exponential decay is described by the function

$$Q(t) = 2000e^{-0.06t}$$

where t is measured in years, answer the following questions.
a. What is the decay constant?
b. What quantity is present initially?
c. Using a calculator, complete the following table of values:

t	0	5	10	20	100
Q					

3. **GROWTH OF BACTERIA** The growth rate of the bacterium *Escherichia coli,* a common bacterium found in the human intestine, is proportional to its size. Under ideal laboratory conditions, when this bacterium is grown in a nutrient broth medium, the number of cells in a culture doubles approximately every 20 min.
a. If the initial cell population is 100, determine the function $Q(t)$ that expresses the exponential growth of the number of cells of this bacterium as a function of time t (in minutes).
b. How long will it take for a colony of 100 cells to increase to a population of 1 million?
c. If the initial cell population were 1000, how would this alter our model?

4. **WORLD POPULATION** The world population at the beginning of 1990 was 5.3 billion. Assume that the population continues to grow at its present rate of approximately 2%/year and find the function $Q(t)$ that expresses the world population (in billions) as a function of time t (in years) where $t = 0$ corresponds to the beginning of 1990.
a. Using this function, complete the following table of values and sketch the graph of the function Q.

Year	1990	1995	2000	2005
World Population				

Year	2010	2015	2020	2025
World Population				

b. Find the estimated rate of growth in the year 2000.

5. **WORLD POPULATION** Refer to Exercise 4.
a. If the world population continues to grow at its present rate of approximately 2%/year, find the length of time t_0 required for the world population to triple in size.
b. Using the time t_0 found in part (a), what would be the world population if the growth rate were reduced to 1.8%?

6. **RESALE VALUE** A certain piece of machinery was purchased 3 yr ago by the Garland Mills Company for \$500,000. Its present resale value is \$320,000. Assuming that the machine's resale value decreases exponentially, what will it be 4 yr from now?

7. **ATMOSPHERIC PRESSURE** If the temperature is constant, then the atmospheric pressure P (in pounds per square inch) varies with the altitude above sea level h in accordance with the law

$$P = p_0e^{-kh}$$

where p_0 is the atmospheric pressure at sea level and k is a constant. If the atmospheric pressure is 15 lb/in.^{2} at sea level and 12.5 lb/in.^{2} at 4000 ft, find the atmospheric pressure at an altitude of 12,000 ft. How fast is the atmospheric pressure changing with respect to altitude at an altitude of 12,000 ft?

8. **Radioactive Decay** The radioactive element polonium decays according to the law

$$Q(t) = Q_0 \cdot 2^{-(t/140)}$$

where Q_0 is the initial amount and the time t is measured in days. If the amount of polonium left after 280 days is 20 mg, what was the initial amount present?

9. **Radioactive Decay** Phosphorus 32 has a half-life of 14.2 days. If 100 g of this substance are present initially, find the amount present after t days. What amount will be left after 7.1 days? How fast is the phosphorus 32 decaying when $t = 7.1$?

10. **Nuclear Fallout** Strontium 90, a radioactive isotope of strontium, is present in the fallout resulting from nuclear explosions. It is especially hazardous to animal life, including humans, because, upon ingestion of contaminated food, it is absorbed into the bone structure. Its half-life is 27 yr. If the amount of strontium 90 in a certain area is found to be four times the "safe" level, find how much time must elapse before an "acceptable level" is reached.

11. **Carbon-14 Dating** Wood deposits recovered from an archeological site contain 20% of the carbon 14 they originally contained. How long ago did the tree from which the wood was obtained die?

12. **Carbon-14 Dating** Skeletal remains of the so-called "Pittsburgh Man," unearthed in Pennsylvania, had lost 82% of the carbon 14 they originally contained. Determine the approximate age of the bones.

13. **Learning Curves** The American Court Reporting Institute finds that the average student taking Advanced Machine Shorthand, an intensive 20-wk course, progresses according to the function

$$Q(t) = 120(1 - e^{-0.05t}) + 60 \qquad (0 \le t \le 20)$$

where $Q(t)$ measures the number of words (per minute) of dictation that the student can take in machine shorthand after t wk in the course. Sketch the graph of the function Q and answer the following questions.
a. What is the beginning shorthand speed for the average student in this course?
b. What shorthand speed does the average student attain halfway through the course?
c. How many words per minute can the average student take after completing this course?

14. **Effect of Advertising on Sales** The Metro Department Store found that t wk after the end of a sales promotion the volume of sales was given by a function of the form

$$S(t) = B + Ae^{-kt} \qquad (0 \le t \le 4)$$

where $B = 50{,}000$ and is equal to the average weekly volume of sales before the promotion. The sales volumes at the end of the first and third weeks were \$83,515 and \$65,055, respectively. Assume that the sales volume is decreasing exponentially.
a. Find the decay constant k.
b. Find the sales volume at the end of the fourth week.
c. How fast is the sales volume dropping at the end of the fourth week?

15. **Demand for Computers** The Universal Instruments Company found that the monthly demand for its new line of Galaxy Home Computers t mo after placing the line on the market was given by

$$D(t) = 2000 - 1500e^{-0.05t} \qquad (t > 0)$$

Graph this function and answer the following questions.
a. What is the demand after 1 mo? After 1 yr? After 2 yr? After 5 yr?
b. At what level is the demand expected to stabilize?
c. Find the rate of growth of the demand after the tenth month.

16. **Newton's Law of Cooling** Newton's law of cooling states that the rate at which the temperature of an object changes is proportional to the difference in temperature between the object and that of the surrounding medium. Thus, the temperature $F(t)$ of an object that is greater than the temperature of its surrounding medium is given by

$$F(t) = T + Ae^{-kt}$$

where t is the time expressed in minutes, T is the temperature of the surrounding medium, and A and k are constants. Suppose a cup of instant coffee is prepared with boiling water (212°F) and left to cool on the counter in a room where the temperature is 72°F. If $k = 0.1865$, determine when the coffee will be cool enough to drink (say, 110°F).

17. **Spread of an Epidemic** During a flu epidemic, the number of children in the Woodbridge Community School System who contracted influenza after t days was given by

$$Q(t) = \frac{1000}{1 + 199e^{-0.8t}}$$

a. How many children were stricken by the flu after the first day?

b. How many children had the flu after 10 days?
c. How many children eventually contracted the disease?

18. Growth of a Fruit-Fly Population On the basis of data collected during an experiment, a biologist found that the growth of the fruit fly (*Drosophila*) with a limited food supply could be approximated by the exponential model

$$N(t) = \frac{400}{1 + 39e^{-0.16t}}$$

where t denotes the number of days since the beginning of the experiment.
a. What was the initial fruit-fly population in the experiment?
b. What was the maximum fruit-fly population that could be expected under this laboratory condition?
c. What was the population of the fruit-fly colony on the 20th day?
d. How fast was the population changing on the 20th day?

19. Percentage of Households with VCRs According to estimates by Paul Kroger Associates, the percentage of households that own videocassette recorders (VCRs) is given by

$$P(t) = \frac{68}{1 + 21.67e^{-0.62t}} \qquad (0 \le t \le 12)$$

where t is measured in years, with $t = 0$ corresponding to the beginning of 1985. What percentage of households owned VCRs at the beginning of 1985? At the beginning of 1995?

20. Population Growth in the Twenty-First Century The U.S. population is approximated by the function

$$P(t) = \frac{616.5}{1 + 4.02e^{-0.5t}}$$

where $P(t)$ is measured in millions of people and t is measured in 30-yr intervals, with $t = 0$ corresponding to 1930. What is the expected population of the United States in 2020 ($t = 3$)?

21. Spread of a Rumor Three hundred students attended the dedication ceremony of a new building on a college campus. The president of the traditionally female college announced a new expansion program, which included plans to make the college coeducational. The number of students who learned of the new program t hr later is given by the function

$$f(t) = \frac{3000}{1 + Be^{-kt}}$$

If 600 students on campus had heard about the new program 2 hr after the ceremony, how many students had heard about the policy after 4 hr? How fast was the rumor spreading 4 hr after the ceremony?

22. Chemical Mixtures Two chemicals react to form another chemical. Suppose the amount of the chemical formed in time t (in hours) is given by

$$x(t) = \frac{15\left[1 - \left(\frac{2}{3}\right)^{3t}\right]}{1 - \frac{1}{4}\left(\frac{2}{3}\right)^{3t}}$$

where $x(t)$ is measured in pounds. How many pounds of the chemical are formed eventually?
Hint: You need to evaluate $\lim_{t\to\infty} x(t)$.

23. Concentration of Glucose in the Bloodstream A glucose solution is administered intravenously into the bloodstream at a constant rate of r mg/hr. As the glucose is being administered, it is converted into other substances and removed from the bloodstream. Suppose the concentration of the glucose solution at time t is given by

$$C(t) = \frac{r}{k} - \left[\left(\frac{r}{k}\right) - C_0\right]e^{-kt}$$

where C_0 is the concentration at time $t = 0$ and k is a constant.
a. Assuming that $C_0 < r/k$, evaluate

$$\lim_{t\to\infty} C(t)$$

and interpret your result.
b. Sketch the graph of the function C.

24. Gompertz Growth Curve Consider the function

$$Q(t) = Ce^{-Ae^{-kt}}$$

where $Q(t)$ is the size of a quantity at time t and A, C, and k are positive constants. The graph of this function, called the **Gompertz growth curve,** is used by biologists to describe restricted population growth.
a. Show that the function Q is always increasing.
b. Find the time t at which the growth rate $Q'(t)$ is increasing most rapidly.
Hint: Find the inflection point of Q.
c. Show that $\lim_{t\to\infty} Q(t) = C$ and interpret your result.

Solution to Self-Check Exercise 13.5

We are given that $P_0 = 226.5$, $k = 0.008$, and $I = 0.5$. So

$$P = \left(226.5 + \frac{0.5}{0.008}\right)e^{0.008t} - \frac{0.5}{0.008}$$
$$= 289e^{0.008t} - 62.5$$

Therefore, the population in the year 2005 will be given by

$$P(25) = 289e^{0.2} - 62.5$$
$$\approx 290.5$$

or approximately 290.5 million.

Chapter 13 Summary of Principal Formulas and Terms

Formulas

1. Exponential function with base b	$y = b^x$
2. The number e	$e = \lim\limits_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = 2.71828$
3. Exponential function with base e	$y = e^x$
4. Logarithmic function with base b	$y = \log_b x$
5. Logarithmic function with base e	$y = \ln x$
6. Inverse properties of $\ln x$ and e	$\ln e^x = x$ and $e^{\ln x} = x$
7. Continuous compound interest	$A = Pe^{rt}$
8. Derivative of the exponential function	$\frac{d}{dx}(e^x) = e^x$
9. Chain rule for exponential functions	$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$
10. Derivative of the logarithmic function	$\frac{d}{dx} \ln \lvert x \rvert = \frac{1}{x}$
11. Chain rule for logarithmic functions	$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

Terms

common logarithm
natural logarithm
logarithmic differentiation
exponential growth
growth constant
exponential decay
decay constant
half-life of a radioactive element
logistic growth function

CHAPTER 13 REVIEW EXERCISES

1. Sketch on the same set of coordinate axes the graphs of the exponential functions defined by the equations.
   a. $y = 2^{-x}$ **b.** $y = \left(\frac{1}{2}\right)^x$

In Exercises 2 and 3, express each in logarithmic form.

2. $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$ 3. $16^{-3/4} = 0.125$

In Exercises 4 and 5, solve each equation for *x*.

4. $\log_4(2x + 1) = 2$
5. $\ln(x - 1) + \ln 4 = \ln(2x + 4) - \ln 2$

In Exercises 6–8, given that ln 2 = *x*, ln 3 = *y*, and ln 5 = *z*, express each of the given logarithmic values in terms of *x*, *y*, and *z*.

6. ln 30 7. ln 3.6 8. ln 75
9. Sketch the graph of the function $y = \log_2(x + 3)$.
10. Sketch the graph of the function $y = \log_3(x + 1)$.

In Exercises 11–28, find the derivative of the function.

11. $f(x) = xe^{2x}$
12. $f(t) = \sqrt{t}e^t + t$
13. $g(t) = \sqrt{t}e^{-2t}$
14. $g(x) = e^x\sqrt{1 + x^2}$
15. $y = \dfrac{e^{2x}}{1 + e^{-2x}}$
16. $f(x) = e^{2x^2-1}$
17. $f(x) = xe^{-x^2}$
18. $g(x) = (1 + e^{2x})^{3/2}$
19. $f(x) = x^2e^x + e^x$
20. $g(t) = t \ln t$
21. $f(x) = \ln(e^{x^2} + 1)$
22. $f(x) = \dfrac{x}{\ln x}$
23. $f(x) = \dfrac{\ln x}{x + 1}$
24. $y = (x + 1)e^x$
25. $y = \ln(e^{4x} + 3)$
26. $f(r) = \dfrac{re^r}{1 + r^2}$
27. $f(x) = \dfrac{\ln x}{1 + e^x}$
28. $g(x) = \dfrac{e^{x^2}}{1 + \ln x}$
29. Find the second derivative of the function $y = \ln(3x + 1)$.
30. Find the second derivative of the function $y = x \ln x$.
31. Find $h'(0)$ if $h(x) = g(f(x))$, $g(x) = x + (1/x)$, and $f(x) = e^x$.
32. Find $h'(1)$ if $h(x) = g(f(x))$, $g(x) = \dfrac{x + 1}{x - 1}$, and $f(x) = \ln x$.
33. Use logarithmic differentiation to find the derivative of $f(x) = (2x^3 + 1)(x^2 + 2)^3$.
34. Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{x(x^2 - 2)^2}{(x - 1)}$.
35. Find an equation of the tangent line to the graph of $y = e^{-2x}$ at the point $(1, e^{-2})$.
36. Find an equation of the tangent line to the graph of $y = xe^{-x}$ at the point $(1, e^{-1})$.
37. Sketch the graph of the function $f(x) = xe^{-2x}$.
38. Sketch the graph of the function $f(x) = x^2 - \ln x$.
39. Find the absolute extrema of the function $f(t) = te^{-t}$.
40. Find the absolute extrema of the function

$$g(t) = \frac{\ln t}{t}$$

on [1, 2].
41. A hotel was purchased by a conglomerate for \$4.5 million and sold 5 yr later for \$8.2 million. Find the annual rate of return (compounded continuously).
42. Find the present value of \$119,346 due in 4 yr at an interest rate of 10%/year compounded continuously.
43. A culture of bacteria that initially contained 2000 bacteria has a count of 18,000 bacteria after 2 hr.
    a. Determine the function $Q(t)$ that expresses the exponential growth of the number of cells of this bacterium as a function of time t (in minutes).
    b. Find the number of bacteria present after 4 hr.
44. The radioactive element radium has a half-life of 1600 yr. What is its decay constant?
45. The VCA Television Company found that the monthly demand for its new line of video disc players t mo after placing the players on the market is given by

$$D(t) = 4000 - 3000e^{-0.06t} \qquad (t \geq 0)$$

Graph this function and answer the following questions.
    a. What was the demand after 1 mo? After 1 yr? After 2 yr?
    b. At what level is the demand expected to stabilize?
46. During a flu epidemic, the number of students at a certain university who contracted influenza after t days could be approximated by the exponential model

$$Q(t) = \frac{3000}{1 + 499e^{-kt}}$$

If 90 students contracted the flu by day 10, how many students contracted the flu by day 20?