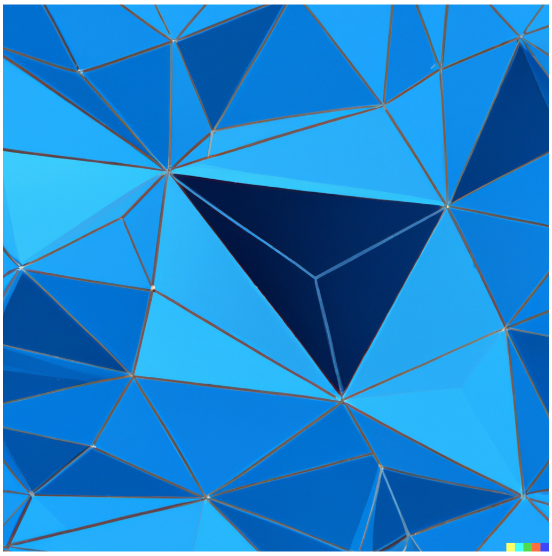
Universidad de Guanajuato

Departamento de Matemáticas

Solving Differential Equations with Discrete Exterior Calculus



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Discrete Exterior Calculus (DEC) aim to discretize Exterior Differential Calculus. Definitions in (DEC) tries to mimic the existing smooth theory [4]. In his doctoral thesis [6], in the year 2023, Hirani proposed fundaments of Discrete Exterior Calculus using Discrete Combinatorics and Algebraic Topology definitions. In this way, discrete tools were proposed in order to have an equivalent to differential forms and operators, vector fields, geometric operators, etcetera.

One of main DEC applications is the creation of discrete operators to solve Partial Differential Equations (PDE's) numerically. The proof of it is given in [5] in the year of 2008 where authors discretize Darcy flow equations in order to solve them numerically.

In this project made for a Partial Differential Equations course an introduction to basic concepts to DEC is presented trying not to avoid theoretical definitions such as the ones from Algebraic Topology. On the other hand, such theoretical aspects are widely explained through drawings, figures and examples.

Section 1 explain definitions related to simplicial complexes borrowed from Algebraic Topology. Section 2 introduce to Discrete Exterior Calculus explaining steps to discretize PDE's. Finally Section 3 presents three DEC applications. Specifically Poisson equations with Dirichlet conditions in $\mathbb{R}$ and $\mathbb{R}^2$ and a Heat equation example.

Contents

1 Simplicial Complexes

This section presents some definitions of Algebraic Topology that will be useful for our purposes. Most definitions here, are taken from [8] and [9].

1.1 Euclidean Simplicial Complexes

We begin by defining what a k-complex is. First, let us remember some Linear Algebra definitions. An *affine subspace* of a vector space V is the set $f^{-1}(0)$ where $f : V \to W$ is a function of the form $f(x) = T(x) + b$ where, W is a vector space and T is a linear transformation $T : V \to W$. The dimension of an affine subspace is defined as the dimension of the kernel: $\ker(T)$. Finally, $(k-1)$-dimensional affine space of a vector space V, whose dimension is k, is called an *affine hyperplane*.

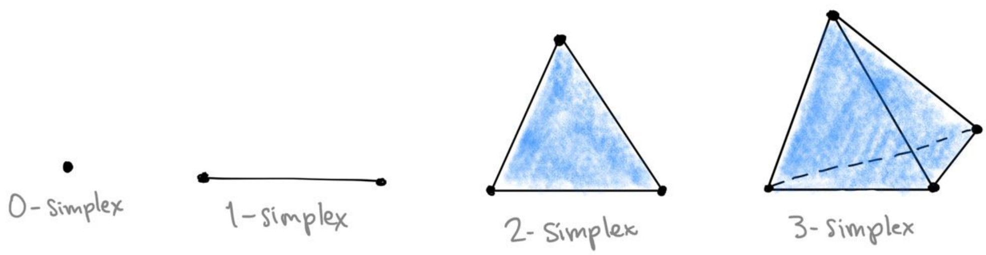
We say that $\{v_0, \ldots, v_k\} \subset \mathbb{R}^n$ are in *general position* or are *affinely independent* if they are not contained in any $(k-1)$-dimensional affine subspace, or equivalently if $\{v_1 - v_0, \ldots, v_k - v_0\}$ are linearly independent.

Definition 1. A k-simplex denoted by σ is the convex hull of $k+1$ points $v_0, \ldots, v_k \in \mathbb{R}^n$ in general position, i.e

$$\sigma = \left\{ \sum_{i=0}^{k} t_i v_i \in \mathbb{R}^n : \sum_{i=0}^{k} t_i = 1, t_i \geq 0 \right\}.$$

The set $\sigma \subset \mathbb{R}^n$ is given the subspace topology. We also say that $v_0, \ldots, v_k$ are the vertices of the k-simplex and the integer k is the dimension of σ. Notation $\langle v_0, \ldots, v_k \rangle$ denote the simplex spanned by $v_0, \ldots, v_k$.

Let's remember in addition that the convex hull of any subset $A \subset \mathbb{R}^n$ is the intersection of all convex sets containing A. One can express the convex hull, as above, in barycentric coordinates, i.e. as a nonnegative weighted combination of the vertices, where the weights sum up to one. Next image shows a 0, 1, 2 and 3-simplex.



According to our Definition 1, we can notice that we cannot have a m-simplex in $\mathbb{R}^n$ if $m > n$.

If σ is a k-simplex, then:

- we call a simplex spanned by a subset of the vertices of σ a *face*,
- the faces that are different to σ itself are called *proper faces*,

- 0-dimensional and 1-dimensional faces of σ are vertices and edges respectively,
- the union of all proper faces is called the *boundary* of σ and denoted by Bdσ,
- the *interior* of σ is given by $\operatorname{Int}\sigma = \sigma \setminus \operatorname{Bd}\sigma$.

It can be shown that every k-simplex is a k-dimensional manifold with boundary. A n-dimensional manifold with boundary is a second countable Hausdorff space where each point has a neighborhood homeomorphic to an open subset of the n-dimensional *upper half space* $\mathbb{H} := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$. Let's recall that a topological space X is said to be Hausdorff if for every pair of distinct points p_1 and p_2 there exist to disjoint neighborhoods of p_1 and p_2. In the other hand, a topological space X is said to be countable if it has a countable basis.

We desire an estructure that is a bunch of simplices connected or linked nicely in some way:

Definition 2. An Euclidean simplicial complex ia a collection $\mathcal{K}$ of simplices in $\mathbb{R}^n$ such that:

i) If $\sigma \in \mathcal{K}$, then each face of σ is in $\mathcal{K}$.

ii) The intersection of any two simplices in $\mathcal{K}$ is either empty or a face of each.

The dimension of $\mathcal{K}$ is the maximum dimension of any simplex in $\mathcal{K}$.

1.2 Abstract Simplicial Complexes

The notion of Euclidean simplicial complexes can be generalized in order to work with different objects other than subsets of Euclidean spaces. We give a more abstract definition of Simplicial Complexes where we only care about how things are connected to each other and not how they are located geometrically:

Definition 3. An Abstract Simplicial Complex is a collection $\mathcal{K}$ of nonempty finite sets called abstract simplices with the following condition: If $\tau \in \mathcal{K}$, then every nonempty subset of τ is in $\mathcal{K}$. In addition:

- The dimension of an abstract simplex consisting of $k+1$ vertices is defined to be k.
- The dimension of $\mathcal{K}$ is the maximum dimension of any simplex in $\mathcal{K}$, if it exists. Otherwise $\mathcal{K}$ is said to be infinite-dimensional.
- We say that $\mathcal{K}$ is a finite complex if $\mathcal{K}$ is a finite set.
- Any element of a simplex $\tau \in \mathcal{K}$ is called a vertex of τ, and a nonempty subset of τ is called a face of σ.
- A subset of $\mathcal{K}$ that is itself a simplicial complex is called a subcomplex of $\mathcal{K}$.

1.3 Oriented Simplicial Complex

We desire to have the notion of orientation in a k-simplex. We can think a k-simplex as an array of $k+1$ vertices. This array can be sorted in many ways. Having this in mind the following definition is presented in [9].

Definition 4. Let σ be a simplex (either geometric or abstract). Define two orderings of its vertex set to be equivalent if they differ from one another by an even permutation. If $\dim \sigma > 0$, the orderings of the vertices of σ then fall into two equivalence classes. Each of these classes is called an **orientation** of σ. If σ is a 0-simplex, then there is only one class and hence only one orientation of σ. An **oriented simplex** is a simplex σ together with an orientation of σ.

We will use the symbol

$$v_0, \dots, v_k$$

to denote the simplex they span. The symbol

$$[v_0, \dots, v_k]$$

will denote the oriented simplex consisting of the simplex $v_0, \dots, v_k$ and the equivalence class of the particular ordering $(v_0, \dots, v_k)$. An **oriented simplicial complex** is a simplicial complex whose simplices are oriented simplices.

We can compare to simplices by its orientation as it is explained in [2]. If we have a k-simplex complex $\sigma = [v_0, \dots, v_k]$, then the **induced orientation** on each of the $(k-1)$-simplices is given by the equivalence class represented by the simplex $[v_0, \dots, \hat{v}_i, \dots, v_n]$ where $\hat{v}_i$ means v_i is omitted. For example if we have a $2-$simplex $[v_0, v_1, v_2]$ the induced orientation of each of its 1-dimensional faces correspond to $[v_1, v_2]$, $[v_0, v_2]$ and $[v_0, v_1]$.

Now, given two adjacent $k-$simplices (ie they share a maximal face) we can compare their orientations as follows:

Definition 5. Two adjacent oriented simplices have the same **relative orientation** if the (maximal) faces in their intersection have opposite orientation considering both orientations induced by each simplex.

Figure 1.1 presents an easy example in order to understand the notion of relative orientation.

The notion of relative orientation allows to define what is a primal mesh:

Definition 6. A **primal mesh** with dimension n is a simplicial complex with dimension n such that every adjacent n-simplices have the same relative orientation.

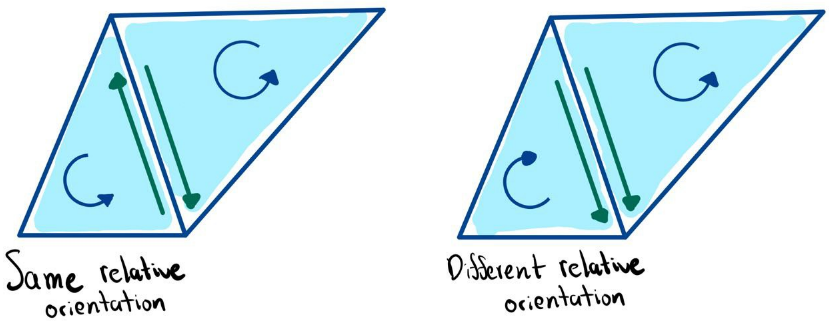


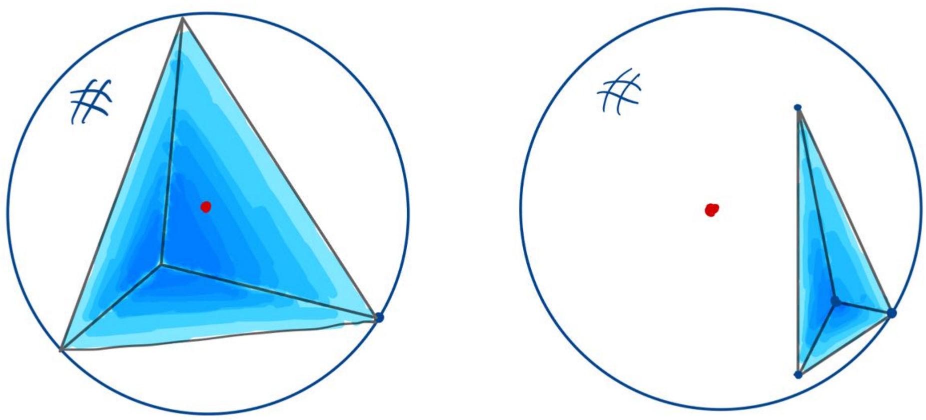
Figure 1.1: No, I didn't swap them.

1.4 Dual Complex of a Simplicial Complex

First, let's remember from elementary school what a circumcenter is. We take the following definition from Hirani's thesis [6] :

Definition 7. The circumcenter of a k-simplex $\sigma \subset \mathbb{R}^k$ denoted by $c(\sigma)$ is given by the center of the unique k-sphere that has all $k+1$ vertices of σ on its surface. If the circumcenter of a simplex lies in its interior we call it a well-centered simplex. A simplicial complex all of whose simplices (of all dimensions) are well-centered will be called a well-centered simplicial complex.

Image below shows on the left a well-centered 3-simplex and on the right a 3-simplex which is not well-centered.



As mention in [6], the circumcenter of a $k-$simplex can be obtained by taking the intersection of the normals to the boundary faces, where the normals are emanating form the circumcenter of the face. In this way one can compute the circumcenter of a k-simplex in a recursive way.

Lets suppose now that we have a simplicial complex $\mathcal{K}$ of dimension n. Somehow we would like to construct a simplicial complex, again of dimension n, starting from the circumcenters of simplices in $\mathcal{K}$. In order to implement this procedure we will need $\mathcal{K}$ to be a well centered simplicial complex, otherwise the circumcenter subdivision may not produce a simplicial complex. Following definition is taken from [6]:

Definition 8. The circumcenter subdivision of a well-centered simplicial complex $\mathcal{K}$ of dimension n is denoted by $\text{csd}(\mathcal{K})$, and it is a simplicial complex with the same underlying space as $\mathcal{K}$ and consisting of all simplices of the form $[c(\sigma_1), \dots, c(\sigma_k)]$ for $k = 1, \dots, n$ (each index of sigma doesn't represent dimensionality). Here $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_k$, which means σ_i is a proper face of σ_j whenever $i < j$ and each σ_i is in $\mathcal{K}$.

The preceding definition may be confusing, so we present a couple of examples. Figure 1.2 shows an example with a simplicial complex with dimension $n = 1$, ie a graph. Figure 1.3 is an example for $n = 2$ consisting only of a single 2-simplex ie a triangle.

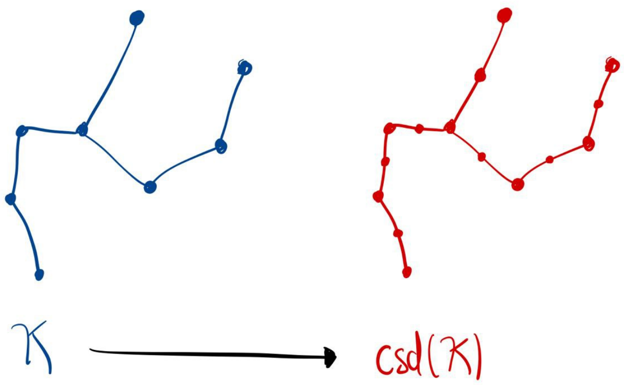


Figure 1.2: On the left it is shown colored in blue the simplicial complex $\mathcal{K} = (V, E)$. The 0-simplices of $\text{csd}(\mathcal{K})$ (on red) consists of the circumcenters $c(v)$ for each $v \in V$ and $c(e)$ for every $e \in E$, note that $c(e)$ is the midpoint of the edge. In the other hand, 1-simplices of $\text{csd}(\mathcal{K})$ are the two halves of each edge.

This section is concluded with the following definition taken from [6]:

Definition 9. Let $\mathcal{K}$ be a well-centered primal mesh of dimension n and let σ^p be a simplex in $\mathcal{K}$. The **circumcenter dual cell** of σ^p, denoted $D(\sigma^p)$, is given by

$$D(\sigma^p) = \bigcup_{r=0}^{n-p} \bigcup_{\sigma^p \prec \sigma_1 \prec \cdots \prec \sigma_r} \text{Int}\left([c(\sigma^p)c(\sigma_1)\dots c(\sigma_r)]\right).$$

As in Definition 8, $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_k$, means σ_i is a proper face of σ_j whenever $i < j$, and each σ_i is in $\mathcal{K}$. For $r = 0$ interpret $\sigma^p \prec \sigma_1 \prec \dots \prec \sigma_r$ simply as σ^p. Finally, the collection of dual cells is called the **dual cell decomposition** of $\mathcal{K}$. This is a cell complex and will be denoted $D(\mathcal{K})$.

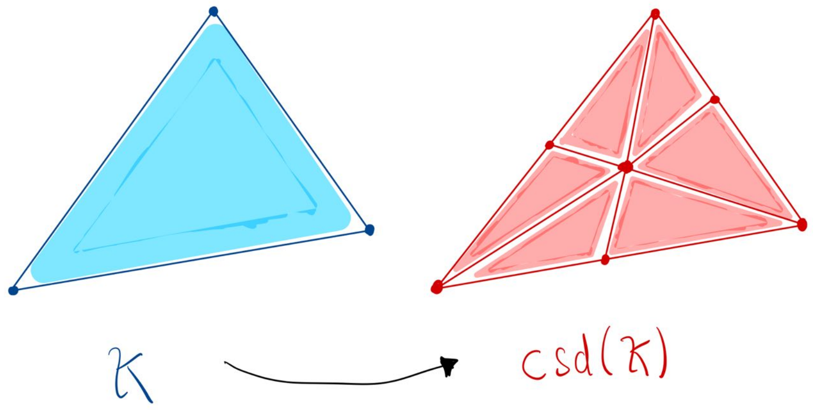
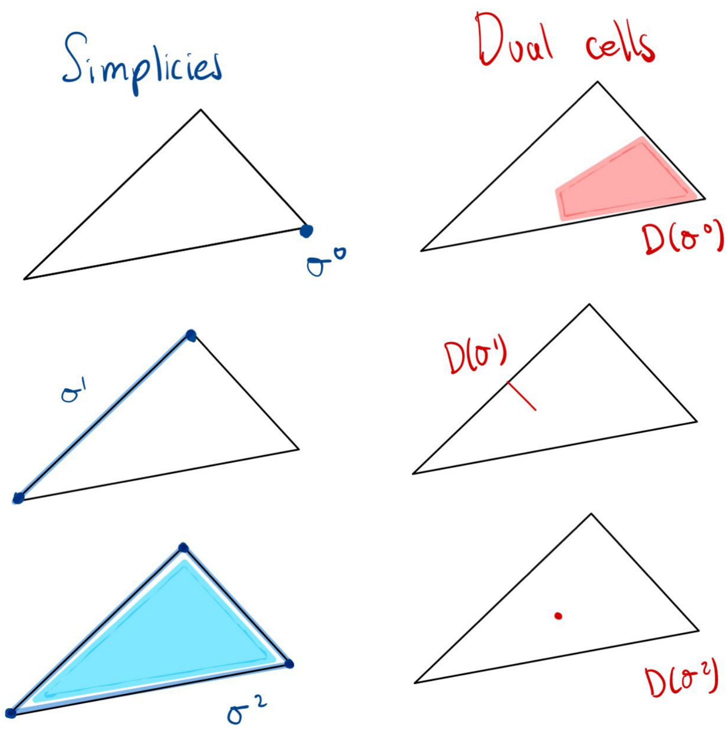


Figure 1.3: On the left it is shown colored in blue the simplicial complex $\mathcal{K}$. The 0-simplices of csd($\mathcal{K}$) (on red) are the circumcenters; each vertex of the triangle, the midpoints of the edges and the circumcenter of the triangle. The 1-simplices are the two halves of each edge and edges joining the circumcenter of the triangle to the vertices and midpoints of the edges.



Definition 9 may be even more confusing that the eighth one. Therefore, an example is shown above. On the right, the circumcenter dual cells corresponding to each simplex on the left.

For practical purposes Figure 1.4 presents how the dual mesh is constructed from the primal mesh. This practical recipe is given in [3]. This will be particularly useful to construct the discrete Hodge star, which at the same time is needed to discretize the Laplacian operator (Sections 2.3 and 2.4).

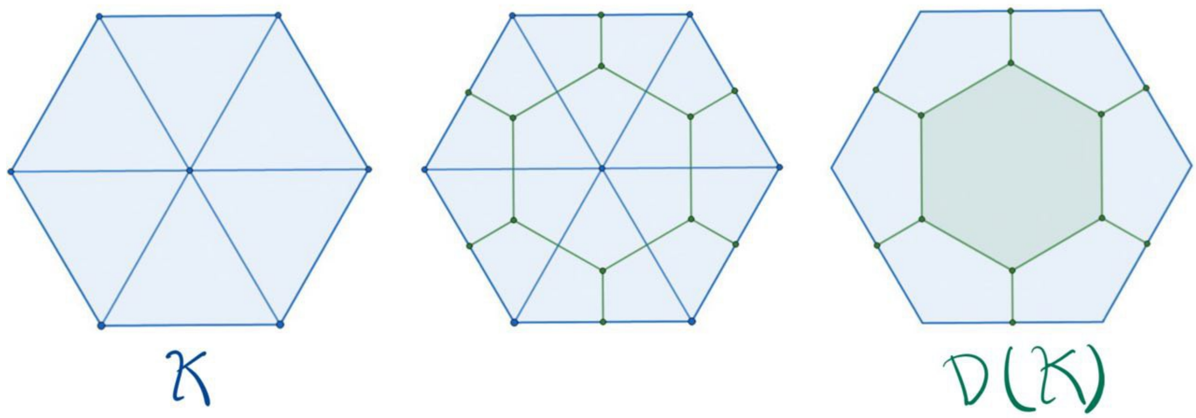


Figure 1.4: Dual mesh construction

2 Discrete Exterior Calculus

2.1 Discrete Differential Forms

We aim to discretize differential forms. The concept of cochain will be introduced here as an analogue of differential forms. At the same time, the role of k-manifolds will be played by k-chains. First, lets understand what a k-chain is with the following definition taken from [9]

Definition 10. Let $\mathcal{K}$ be an oriented simplicial complex. A p-chain on $\mathcal{K}$ is a function c from the set of oriented p-simplices of K, denoted by $\mathcal{K}_p$, to the reals, such that:

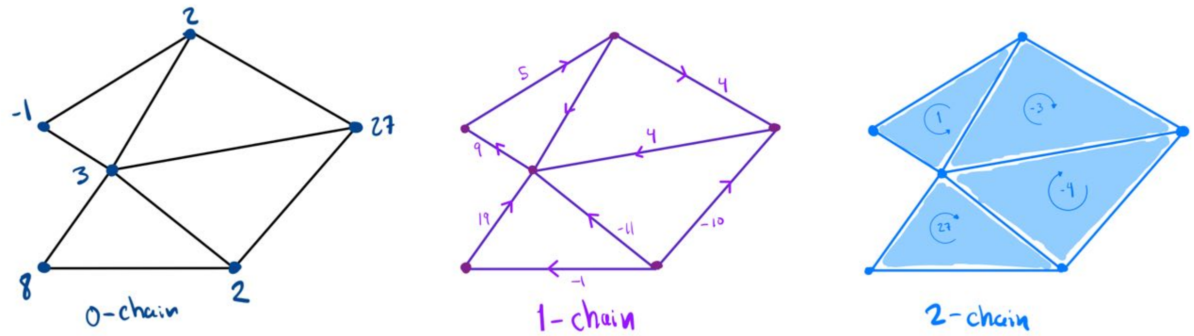
- $c(\sigma) = -c(\sigma')$ if σ and σ' are the opposite orientations of the same simplex.
- $c(\sigma) = 0$ for all but finitely many oriented p-simplices σ.

We add p-chains by adding their values; the resulting group is denoted $C_p(\mathcal{K})$ and its called the group of (oriented) p-chains of K. If $p < 0$ or $p > \dim \mathcal{K})$, we let $C_p(\mathcal{K})$ denote the trivial group.

Important remark: Usually p-chains are considered to be functions $c : \mathcal{K}_p \to \mathbb{Z}$. As mentioned in [6] it is possible to define p-chains as functions $c : \mathcal{K}_p \to \mathbb{R}$. This has the advantage that $C_p(\mathcal{K})$ is not only a group, but also a vector space over $\mathbb{R}$ with dimension $|\mathcal{K}_p|$. This, according to Hirani, has some advantages. Actually in [4], p-chains are defined in this way. **We will consider p-chains as definition above.**

As previous definitions, the above one seems too abstract as well. Nevertheless, as remarked in we can think about a p-chain simply as an array or table of the oriented $p-$simplices of the given complex $\mathcal{K}$. An integer is entered corresponding to each simplex. We can add up to tables by adding up the corresponding entries. Such set of tables turns out to be an abelian group.

The following figure shows an example of a 0-chain, 1-chain and 2-chain over the same simplicial complex of dimension $n = 2$. Here c only takes integers values, but as mentioned in the last remark, it could take any real values.



Now we will define a particular chain that will result to be very useful

Definition 11. If σ is an oriented simplex, the **elementary chain** c corresponding to σ is the function defined as follows:

$$\begin{aligned} c(\sigma) &= 1, \\ c(\sigma') &= -1, \quad \text{if } \sigma' \text{ is the opposite orientation of } \sigma, \\ c(\tau) &= 0 \quad \text{for all other oriented simplices.} \end{aligned}$$

Above definition was taken from [9]. For an oriented simplex σ, the elementary chain c corresponding to σ will be denoted by c_σ. It is worth to mention that often the symbol σ is used not only to denote a simplex but also to denote the corresponding chain. Therefore, with this convention you can write $\sigma = -\sigma'$. However we will try to avoid such convention and we'll use c_σ instead.

We can observe that, given a simplicial complex $\mathcal{K}$, the set of elementary chains corresponding to each simplex in $\mathcal{K}_p$ (considering only one orientation of each simplex) is a basis for $C_p(\mathcal{K})$. Such basis will be denoted by α_p.

Now we can define our analogue to differential forms whose role will be played, as it was already mention, by $p-$cochains. We'll consider definition presented in [4]:

Definition 12. Given a simplicial complex $\mathcal{K}$. A $p-$cochain is a linear mapping $\omega : C_p(\mathcal{K}) \longrightarrow \mathbb{R}$.

Let's remember that $C_p(\mathcal{K})$ is a vector space whose dimension is $|\mathcal{K}_p|$. Then the space of p-cochains is also a vector space with dimension $|\mathcal{K}_p|$. Therefore, if we represent each cochains and chains as column vectors, we use following notations

$$\omega(c) = \omega^T c = \langle \omega, c \rangle \quad \text{for each} \quad c \in C_p(\mathcal{K}).$$

We connect this subsection with the next one through following definition taken again from [4]. This definition highlight the analogy of chains and cochains with manifolds and forms respectively:

Definition 13. The integral of a $p-$cochain ω over a $p-$chain c is defined to be

$$\int_c \omega = \omega(c).$$

Putting together notations we have for a p-chain c and a p-cochain ω that:

$$\int_c \omega = \omega(c) = \omega^T c = \langle \omega, c \rangle .$$

2.2 Discrete Exterior Derivative

Here we aim to define an analogue version of the exterior derivative. First we need to define the boundary operator ∂. Definition below is taken from [9]

Definition 14. Let $\mathcal{K}$ be a simplicial complex. The **boundary operator** is a homomorphism:

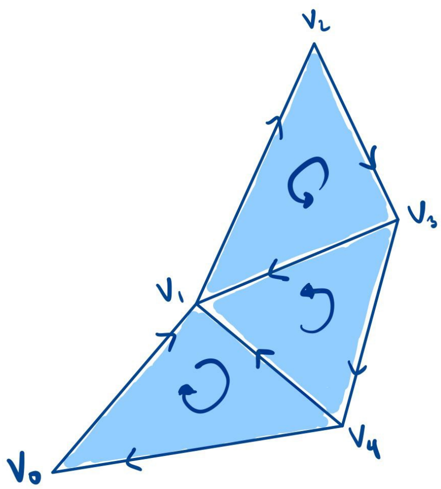
$$\partial_p : C_p(\mathcal{K}) \to C_{p-1}(\mathcal{K}).$$

If $\sigma = [v_0, \ldots, v_p]$ is an oriented simplex with $p > 0$, we define

$$\partial_p(c_\sigma) = \partial_p(c_{[v_0,\ldots,v_p]}) = \sum_{i=0}^{p} (-1)^i c_{[v_0,\ldots,\hat{v}_i,\ldots,v_p]},$$

where the symbol $\hat{v}_i$ means that the vertex v_i is to be deleted from the array. Since $C_p(\mathcal{K})$ is the trivial group for $p < 0$, the operator ∂_p is the trivial homomorphism for $p \leq 0$.

We observe that the boundary operator ∂_p is actually a linear transformation. Therefore ∂_p can be represented as a matrix of size $|\mathcal{K}_{p-1}| \times |\mathcal{K}_p|$. As in previous definitions an example is presented. A very similar example with a hexagon is explained in [3]. Lets consider the following simplicial complex $\mathcal{K}$:



In this way, for the simplicial complex $\mathcal{K}$ above we consider the following ordered basis for $C_0(\mathcal{K})$, $C_1(\mathcal{K})$ and $C_2(\mathcal{K})$

- 0-simplices:

$$\alpha_0 = \left\{ c_{[v_0]}, c_{[v_1]}, c_{[v_2]}, c_{[v_3]}, c_{[v_4]} \right\} \subset C_0(\mathcal{K}).$$

- 1-simplices:

$$\alpha_1 = \left\{ c_{[v_0,v_1]}, c_{[v_1,v_2]}, c_{[v_2,v_3]}, c_{[v_3,v_4]}, c_{[v_4,v_0]}, c_{[v_4,v_1]}, c_{[v_3,v_1]} \right\} \subset C_1(\mathcal{K}).$$

- 2-simplices:

$$\alpha_2 = \left\{ c_{[v_0,v_1,v_4]}, c_{[v_4,v_3,v_1]}, c_{[v_1,v_3,v_2]} \right\} \subset C_2(\mathcal{K}).$$

We compute ∂_1 for each edge:

$$\begin{aligned}
\partial_1(c_{[v_0,v_1]}) &= c_{[v_1]} - c_{[v_0]}, \\
\partial_1(c_{[v_1,v_2]}) &= c_{[v_2]} - c_{[v_1]}, \\
\partial_1(c_{[v_2,v_3]}) &= c_{[v_3]} - c_{[v_2]}, \\
\partial_1(c_{[v_3,v_4]}) &= c_{[v_4]} - c_{[v_3]}, \\
\partial_1(c_{[v_4,v_0]}) &= c_{[v_0]} - c_{[v_4]}, \\
\partial_1(c_{[v_4,v_1]}) &= c_{[v_1]} - c_{[v_4]}, \\
\partial_1(c_{[v_3,v_1]}) &= c_{[v_1]} - c_{[v_3]}.
\end{aligned}$$

Then the **matrix representation of ∂_1 in the ordered basis α_1 and α_0 is**

$$[\partial]_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix}$$

We observe that ∂_1 is actually the operator $D^T = -\text{div}$ on a graph as seen in the first part of the EDP course [1] in the Discrete Calculus section.

Now we compute ∂_2 for each triangle:

$$\begin{aligned}
\partial_2(c_{[v_0,v_1,v_4]}) &= c_{[v_1,v_4]} - c_{[v_0,v_4]} + c_{[v_0,v_1]} = c_{[v_0,v_1]} + c_{[v_4,v_0]} - c_{[v_4,v_1]}, \\
\partial_2(c_{[v_4,v_3,v_1]}) &= c_{[v_3,v_1]} - c_{[v_4,v_1]} + c_{[v_4,v_3]} = -c_{[v_3,v_4]} - c_{[v_4,v_1]} + v_{[v_3,v_1]}, \\
\partial_2(c_{[v_1,v_3,v_2]}) &= c_{[v_3,v_2]} - c_{[v_1,v_2]} + c_{[v_1,v_3]} = -c_{[v_1,v_2]} - c_{[v_2,v_3]} - c_{[v_3,v_1]}.
\end{aligned}$$

Then the **matrix representation of ∂_2 in the ordered basis α_2 and α_1 is**

$$[\partial]_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

We can notice that

$$[\partial_1 \circ \partial_2] = [\partial]_1[\partial]_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is not coincidence. It is always satisfied that $\partial_{p-1} \circ \partial_p = 0$ as proved in [9].

We are now ready to discretize the exterior derivative operator:

Definition 15. The pth discrete exterior derivative of ω is the transpose of the $(p+1)$st boundary operator:

$$d_p = \partial_{p+1}^T.$$

Therefore d_p can be represented by a $|\mathcal{K}_{p+1}| \times |\mathcal{K}_p|$.

The justification of previous definition is that, considering Definition 13, we can count on the Stokes Theorem in a discrete setting:

$$\int_c d_p\omega = \langle d_p\omega, c\rangle = \left\langle \partial_{p+1}^T \omega, c\right\rangle = \langle \omega, \partial_{p+1} c\rangle = \langle \partial_{p+1} c, \omega\rangle = \int_{\partial_{p+1} c} \omega.$$

where c is a p-chain and ω is a p-cochain.

2.3 The Hodge Star

In this subsection we aim to discretize Hodge star operator in $\mathbb{R}^2$. First we need to understand how Hodge star operator exactly works, and overall, what exactly is the Hodge star operator. We'll focus in $\mathbb{R}^2$ and everything well be tried to be explained in a simplified and not too theoretical way as it is done in [3] and [7].

2.3.1 Hodge Star in Exterior Algebra

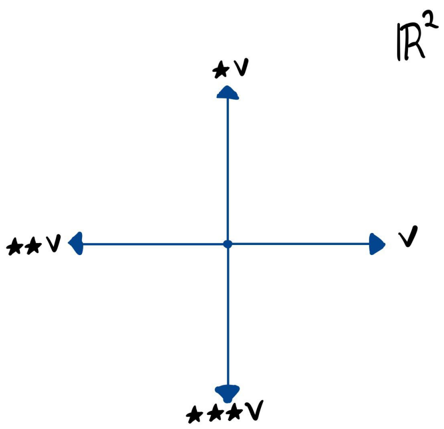
In exterior algebra, the Hodge star operator, denoted by $\star$, is an analogue of orthogonal complement in linear algebra. In general, if v is a k-vector $\in \mathbb{R}^n$, then $\star v$ is a $(n-k)$-vector that somehow is a complement in some way [7].

Let's consider the canonical vector $e_1 \in \mathbb{R}^2$. We aim to find another vector $v \in \mathbb{R}^2$ such that they form has area 1. There are multiple choices for v, however, if we desire orthogonality and standard orientation the unique solution is the canonical vector $v = e_2$. This is what $\star$ operator does, it says that e_1 and e_2 are complement between them. This is, $\star e_1 = e_2$ and $\star e_2 = -e_1$.

The equation that defines the Hodge star operator for any $v \in \mathbb{R}^2$ is

$$w \wedge (\star v) = \langle w, v\rangle\, e_1 \wedge e_2 \quad \text{for every} \quad w \in \mathbb{R}^2.$$

Particularly $v \wedge (\star v) = |v|^2\, e_1 \wedge e_2$ is satisfied. This means v and $\star v$ form a square whose area is $|v|^2$. Therefore, for vectors in $\mathbb{R}^2$, the Hodge star is just a quarter-rotation in the counter-clockwise direction as shown below.



We've seen what $\star$ does to vectors in $\mathbb{R}^2$. Now, let's see what $\star$ does to bivectors. This, is let's see what $\star(v \wedge w)$ is for $v, w \in \mathbb{R}^2$. We need to treat bivectors as vectors not in $\mathbb{R}^2$ but in a different space: $\bigwedge^2 \mathbb{R}^2$. Thus, somehow replicating relation above for vectors in $\mathbb{R}^2$, we have:

$$(v \wedge w) \wedge \star(v \wedge w) = \mathrm{vol}(v \wedge w)^2 e_1 \wedge e_2$$

where $\mathrm{vol}(v \wedge w)$ is the area of $v \wedge w$. We know that $v \wedge w = \mathrm{vol}(v \wedge w) e_1 \wedge e_2$ is a bivector. Therefore, $\star(v \wedge w)$ has to be a vector, specifically

$$\star(v \wedge w) = \mathrm{vol}(v \wedge w).$$

We can notice that $\star(e_1 \wedge e_2) = \mathrm{vol}(e_1 \wedge e_2) = 1$.

Although explanation above is enough for our purposes, the formal definition of the Hodge star operator is presented:

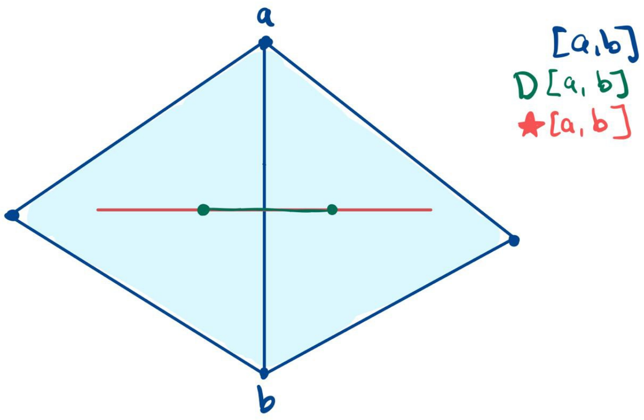
Definition 16. Let V be a $n-$dimensional vector space with a symmetric bilinear form $\langle \cdot, \cdot \rangle$ and the orthonormal oriented basis $\{e_1, \ldots, e_n\}$. The **Hodge star operator** is a linear operator mapping k-vectors to $(n-k)-$vectors $(k = 1, \ldots, n)$ with the following property which defines it completely:

$$\alpha \wedge (\star\beta) = \left\langle \alpha, \beta \right\rangle e_1 \wedge e_2 \wedge \cdots \wedge e_n.$$

2.3.2 Discrete Hodge Star

We saw above that if v is a $k-$vector, then $\star v$ is a $(n-k)-$vector. In the other hand, in Section 1.4 we observed that the dual cell of a k-simplex has dimension $n-k$ (see image below Definition 9). This gives a clue: dual meshes will play some important role that will help discretize the Hodge star operator.

We'd said that in $\mathbb{R}^2$, Hodge star operator only rotates vectors 90 degrees in the counter-clockwise direction. If we have an edge $[a, b]$ we get $\star[a, b]$ by rotating $[a, b]$ 90 degrees around its midpoint. We can see that the dual cell $D[a, b]$ is in the same position than $\star[a, b]$ but it has different length.



However we know that

$$\ell(\star[a,b]) = \ell([a,b])$$

where ℓ represents the length of an edge. Therefore we have the relationship:

$$\frac{D[a,b]}{\ell(D[a,b])} = \frac{\star[a,b]}{\ell(\star[a,b])} = \frac{\star[a,b]}{\ell([a,b])}.$$

Thus

$$D[a,b] = \frac{\ell(D[a,b])}{\ell([a,b])} \star [a,b]$$

If we apply Hodge operator to every 1-simplex (edges) we get the diagonal **Discrete Hodge matrix**:

$$\mathrm{M}_1 = \begin{pmatrix} \frac{\ell(D(e_1))}{\ell(e_1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\ell(D(e_2))}{\ell(e_2)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\ell(D(e_2))}{\ell(e_2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{\ell(D(e_m))}{\ell(e_m)} \end{pmatrix}$$

where

- e_i is each 1-simplex (edge) in our primal mesh $\mathcal{K}$ given in some order,
- $D(e_1)$ is the dual cell of e_1, which is again an edge and
- $m = |\mathcal{K}_1|$ is the number of edges in the primal mesh.

In general, let $\{\sigma_i^k\}$ be the k-simplices of K. Then the discrete kth **Hodge star** is a diagonal matrix such that

$$(\mathrm{M}_k)_{ii} = \frac{\mathrm{vol}(D(\sigma_i^k))}{\mathrm{vol}(\sigma_i^k)}$$

where $\mathrm{vol}(\sigma)$ indicates the length, area or volume of σ, which by convention equals one for a point (vertex).

In section 2.1 we saw that evaluating a discrete manifolds (a chain) in a discrete form (a cochain) can be thought as integrating a form over a manifold (cells). This makes sense with our definition of discretize Hodge star: evaluating a discrete form over a primal or dual form should be the same in some way, but we need to take into consideration that primal and dual cells have different volume. Therefore Hodge star normalize by some ratio of lengths, areas or volumes when mapping between primal and dual [7].

We can notice that the inverse of the discrete Hodge star operator takes dual forms to primal forms. Particularly

$$M_0^{-1} = \begin{pmatrix} \frac{1}{\mathrm{vol}(D[v_1])} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\mathrm{vol}(D[v_2])} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\mathrm{vol}(D[v_3])} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{\mathrm{vol}(D[v_m])} \end{pmatrix}$$

will be useful for us in order to discretize the Laplacian operator. Here, v_i is each of the m vertices in the primal mesh.

2.4 Laplacian Operator

Laplace operator is very important because it plays fundamental roles in geometric and physical contexts. For instance, we know that it appears in diffusion and wave equations. We may know the classical definition of the Laplacian, but as we aim to discretize it, we need to write it in a different and convoluted way.

2.4.1 Continuos Laplace Operator

We know that Laplace Operator is defined as follows:

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

for any $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $f \in C^2(\Omega)$. Let's recall that the divergence operator is given by

$$\mathrm{div}\, F = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

where $F = (F_1, \ldots, F_n)$ is a differentiable vector field. We can write Laplace operator as

$$\Delta f = \mathrm{div}\,(\mathrm{grad}\, f).$$

Such expression is not enough to discretize Δ.

We'll focus only in $\Omega \subset \mathbb{R}^2$. Let $f : \Omega \to \mathbb{R}$ a function in $C^2(\Omega)$. If we apply the exterior derivative to f we get a 1-form:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

We can encode a 1-form in a vector field using the sharp operator $\sharp$, flat operator $\flat$ does the reverse. Then we have the vector field:

$$\mathrm{grad}\, f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (df)^{\sharp}.$$

Let's keep that in mind and procede to discretize div operator. Let $F = (F_1, F_2)$ be a differentiable vector field. We know that applying the Hodge star operator to a 1-form in $\mathbb{R}^2$ is the same as rotating the 1-form by 90 degrees counter-clockwise. This is,

$$\star(A\,dx + B\,dy) = -B\,dx + A\,dy.$$

for any 1-form in $\mathbb{R}^2$. Therefore

$$\begin{aligned}\star F^\flat &= \star(F_1\,dx + F_2\,dy)\\ &= -F_2\,dx + F_1\,dy.\end{aligned}$$

Applying exterior derivative we get:

$$\begin{aligned}d\star F^\flat &= d(-F_2\,dx + F_1\,dy)\\ &= d(-F_2\,dx) + d(F_1\,dy)\\ &= -\left(\frac{\partial F_2}{\partial x}\,dx + \frac{\partial F_2}{\partial y}\,dy\right)\wedge dx + \left(\frac{\partial F_1}{\partial x}\,dx + \frac{\partial F_1}{\partial y}\,dy\right)\wedge dy\\ &= -\frac{\partial F_2}{\partial y}\,dy\wedge dx + \frac{\partial F_1}{\partial x}\,dx\wedge dy\\ &= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right)dx\wedge dy.\end{aligned}$$

By taking the Hodge star in both sides we get the divergence operator.

$$\star d\star F^\flat = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) = \operatorname{div} F.$$

Finally, substituting in above equation $F = \operatorname{grad} f$ and recalling that $\operatorname{grad} f = (df)^\sharp$ we obtain:

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \operatorname{div}((df)^\sharp) = \star d\star((df)^\sharp)^\flat = \star d\star df.$$

We have gotten the desire expression for the Laplacian:

$$\Delta = \star d\star d.$$

A similar computing for $\mathbb{R}^3$ can be found in [7] (chapter 4).

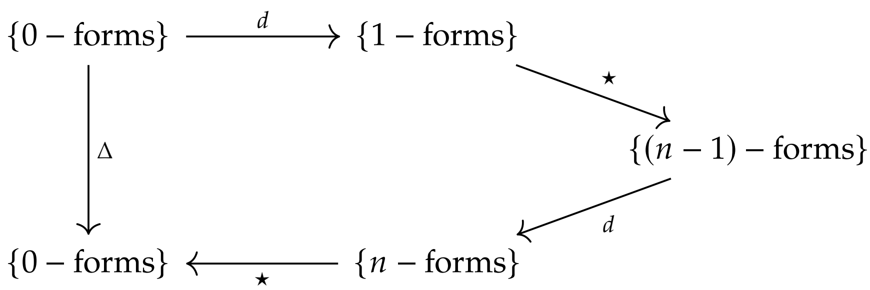


Figure 2.1: Exterior derivative operator takes k-forms to $(k+1)$-forms. Hodge star operator takes k-forms to $(n-k)$-forms. Recall that 0-forms are $\mathbb{R}^n \to \mathbb{R}$ functions.

Laplacian also can be defined for k-forms in general. The discretization for Laplace operator in such case is given by $\Delta = \star d\star d + d\star d\star$. Note that if $k = 0$ the term $d\star d\star$ equals cero (see [7] chapter 6).

2.4.2 Discrete Laplace Operator

We aim to discretize Laplace Operator. We already have done almost all the work as we have discretized Δ in terms of $\star$ and d operators and in addition we have discretized $\star$ and d operators.

Let's suppose that we have a primal mesh $\mathcal{K}$ with N vertices and M edges. We desire to discretize Δf for $f \in C^2(\Omega)$ where $\Omega \subset \mathbb{R}^2$. For each vertex v_i we denote $f_i = f(v_i)$, $i = 1, \ldots, N$. In section 2.1 we define discrete p-forms to be p-cochains. At the same time p-cochains can be represented as vectors (arrays) with length $|\mathcal{K}_p|$. In this case, the vector

$$f = (f_1, f_2, \ldots, f_N)$$

is the discrete version of the 0-form $f : \Omega \to \mathbb{R}$.

In Section 2.2 we define the boundary operator ∂ and the discrete exterior derivate to be $d = \partial^T$. We also saw that ∂_1 can be represented as a $N \times M$ matrix. Therefore $d_0 = \partial_1^T$ is a $M \times N$ matrix. Then $d_0 f$ is a vector in $\mathbb{R}^M$ (array of size M) which can be seen as a discrete 1-form.

Now we have to apply the Hodge star operator M_1 defined in the last subsection. This will take to 1-form $d_0 f$ to another 1-form but in the **dual mesh**. This make sense because $\mathrm{M}_1 d_0 f \in \mathbb{R}^M$ and the dual mesh has M edges as well.

Then, we need to apply another discrete derivate. In this case we can think that operators ∂ and d will switch roles over the dual mesh. Discrete exterior derivative will be in this case the $-\partial_1$ operator. So far we have $-\partial_1 \mathrm{M}_1 d_0 f \in \mathbb{R}^N$ which is a discrete 2-form in the dual mesh.

Let's make a brief parenthesis to connect the ideas with the section of Discrete Calculus in the course. We discretize Laplacian for graphs (simplicial complexes with dimension 1) as $\Delta = \operatorname{div} \operatorname{grad}$. Using notation presented here we have grad $= d_0$ and div $= \partial_1$. However we didn't care about how long the edges were. In this case we do, so we resize calculations via Hodge star operators.

Finally we take another Hodge star operator. Notice that we are in the dual mesh yet, so we need to return to the primal mesh via the **inverse** Hodge operator M_0^{-1} that takes discrete 2-forms in the dual mesh to discrete 0-forms in the primal mesh. Notice that $-\mathrm{M}_0^{-1}\partial_1 \mathrm{M}_1 d_0 f \in \mathbb{R}^N$. The i-th entry of $-\mathrm{M}_0^{-1}\partial_1 \mathrm{M}_1 d_0 f$ takes the value $\Delta f(v_i)$. Avoiding subindices we have the Discrete Laplace operator:

$$\Delta = \mathrm{M}\,\partial \mathrm{M}\,d.$$

Notice that diagram shown in Figure 2.1 also applies to the discrete setting.

3 Solving Differential Equations

3.1 Poisson equation in $\mathbb{R}$

Method explained in this subsection will be useful to solve Poisson equation with boundary Dirichlet conditions

$$\begin{aligned} f''(x) &= g(x), \quad x \in [a, b], \\ f(a) &= \alpha, \quad f(b) = \beta. \end{aligned}$$

where f is the unknown function.

We explain such method with the simple ODE

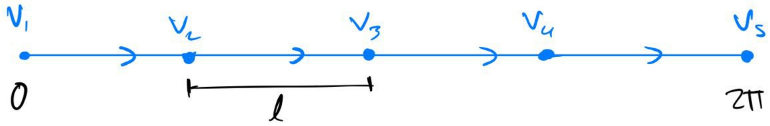
$$f''(x) = \sin(x), \quad x \in [0, 2\pi],$$
$$f(0) = 5, \quad f(2\pi) = 6.$$

By integrating two times in both sides of the equation we get $f(x) = -\sin(x) + c_1 x + c_2$ for constants $c_1, c_2 \in \mathbb{R}$. Boundary condition $f(0) = 5$ implies $c_2 = 5$. Then, $f(2\pi) = 6$ implies $c_1 = 1/2\pi$. Therefore

$$f(x) = -\sin(x) + \frac{1}{2\pi}x + 5.$$

is the ground truth for the solution. Let's approximate such solution through Discrete Exterior Calculus.

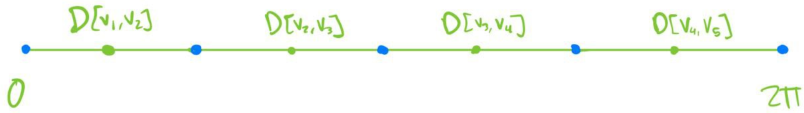
First we discretize interval $[0, 2\pi]$ through a partition. For convenience each subinterval in the partition has same length ℓ.



We need to compute ∂_1 operator for our simplicial complex:

$$\partial_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Recall that $d_0 = \partial_1^T$. We will need Hodge star matrices as well. For that we need the dual mesh



We need M_1 and M_0. In this case, M_1 takes primal 1-forms to dual 0-forms:

$$\mathrm{M}_1 = \begin{pmatrix} \frac{\mathrm{vol}(D[v_1,v_2])}{\mathrm{vol}([v_1,v_2])} & 0 & 0 & 0 \\ 0 & \frac{\mathrm{vol}(D[v_2,v_3])}{\mathrm{vol}([v_2,v_3])} & 0 & 0 \\ 0 & 0 & \frac{\mathrm{vol}(D[v_2,v_3])}{\mathrm{vol}([v_2,v_3])} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{vol}(D[v_3,v_4])}{\mathrm{vol}([v_3,v_4])} \end{pmatrix} = \begin{pmatrix} \frac{1}{\ell} & 0 & 0 & 0 \\ 0 & \frac{1}{\ell} & 0 & 0 \\ 0 & 0 & \frac{1}{\ell} & 1 \\ 0 & 0 & 0 & \frac{1}{\ell} \end{pmatrix}.$$

On the other hand matrix M_0 takes primal 0-forms to dual 1-forms:

$$\mathrm{M}_0 = \begin{pmatrix} \frac{\mathrm{vol}(D[v_1])}{\mathrm{vol}([v_1])} & 0 & 0 & 0 & 0 \\ 0 & \frac{\mathrm{vol}(D[v_2])}{\mathrm{vol}([v_2])} & 0 & 0 & 0 \\ 0 & 0 & \frac{\mathrm{vol}(D[v_3])}{\mathrm{vol}([v_3])} & 0 & 0 \\ 0 & 0 & 0 & \frac{\mathrm{vol}(D[v_4])}{\mathrm{vol}([v_4])} & 0 \\ 0 & 0 & 0 & 0 & \frac{\mathrm{vol}(D[v_5])}{\mathrm{vol}([v_5])} \end{pmatrix} = \begin{pmatrix} \frac{\ell}{2} & 0 & 0 & 0 & 0 \\ 0 & \ell & 0 & 0 & 0 \\ 0 & 0 & \ell & 0 & 0 \\ 0 & 0 & 0 & \ell & 0 \\ 0 & 0 & 0 & 0 & \frac{\ell}{2} \end{pmatrix}.$$

Now, we can construct Discrete Laplacian matrix:

$$\Delta = -\mathrm{M}_0^{-1}\partial_1 \mathrm{M}_1 d_0 = \begin{pmatrix} \frac{-8}{\pi^2} & \frac{8}{\pi^2} & 0 & 0 & 0 \\ \frac{4}{\pi^2} & \frac{-8}{\pi^2} & \frac{4}{\pi^2} & 0 & 0 \\ 0 & \frac{4}{\pi^2} & \frac{-8}{\pi^2} & \frac{4}{\pi^2} & 0 \\ 0 & 0 & \frac{4}{\pi^2} & \frac{-8}{\pi^2} & \frac{4}{\pi^2} \\ 0 & 0 & 0 & \frac{8}{\pi^2} & \frac{-8}{\pi^2} \end{pmatrix}.$$

We have the linear system

$$\Delta f = \begin{pmatrix} \frac{-8}{\pi^2} & \frac{8}{\pi^2} & 0 & 0 & 0 \\ \frac{4}{\pi^2} & \frac{-8}{\pi^2} & \frac{4}{\pi^2} & 0 & 0 \\ 0 & \frac{4}{\pi^2} & \frac{-8}{\pi^2} & \frac{4}{\pi^2} & 0 \\ 0 & 0 & \frac{4}{\pi^2} & \frac{-8}{\pi^2} & \frac{4}{\pi^2} \\ 0 & 0 & 0 & \frac{8}{\pi^2} & \frac{-8}{\pi^2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} = \begin{pmatrix} \sin(v_1) \\ \sin(v_2) \\ \sin(v_3) \\ \sin(v_4) \\ \sin(v_5) \end{pmatrix}$$

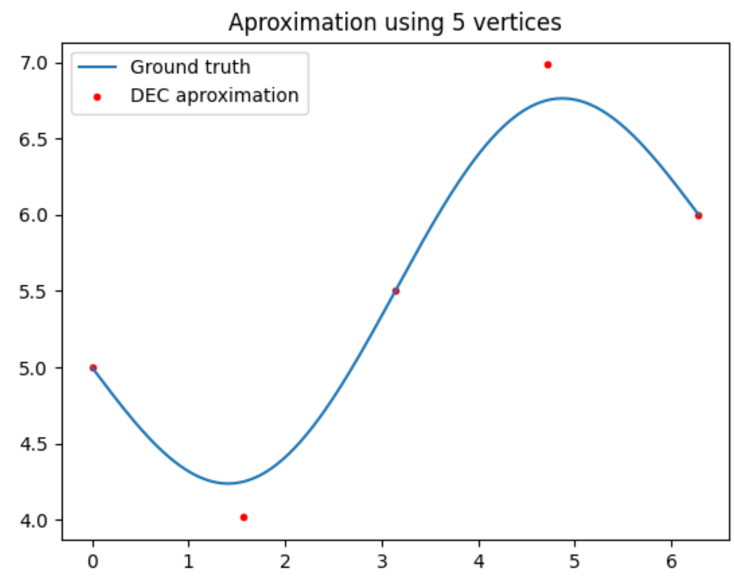
where $f_i := f(v_i)$. We notice that $\mathrm{rank}(\Delta) = 4$. However we know that $f_1 = 5$ and $f_5 = 6$. So we only need to solve the system for f_2, f_3 and f_4. We define $\Delta_{(a,b)}$ as the matrix obtained by deleting rows and columns corresponding to vertices v_1 and v_2 (in this case rows and columns 1 and 5):

$$\Delta_{(0,2\pi)} = \begin{pmatrix} \frac{-8}{\pi^2} & \frac{4}{\pi^2} & 0 \\ \frac{4}{\pi^2} & \frac{-8}{\pi^2} & \frac{4}{\pi^2} \\ 0 & \frac{4}{\pi^2} & \frac{-8}{\pi^2} \end{pmatrix}.$$

Matrix $\Delta_{(0,2\pi)}$ has rank equal to 3 so we can solve the linear system

$$\Delta_{(0,2\pi)} \begin{pmatrix} f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} \sin(v_2) \\ \sin(v_3) \\ \sin(v_4) \end{pmatrix} - \begin{pmatrix} f_1\Delta_{2,1} + f_5\Delta_{2,5} \\ f_1\Delta_{3,1} + f_5\Delta_{3,5} \\ f_1\Delta_{4,1} + f_5\Delta_{4,5} \end{pmatrix}$$

obtaining values f_2, f_3 and f_4. In this way, we get the following approximation

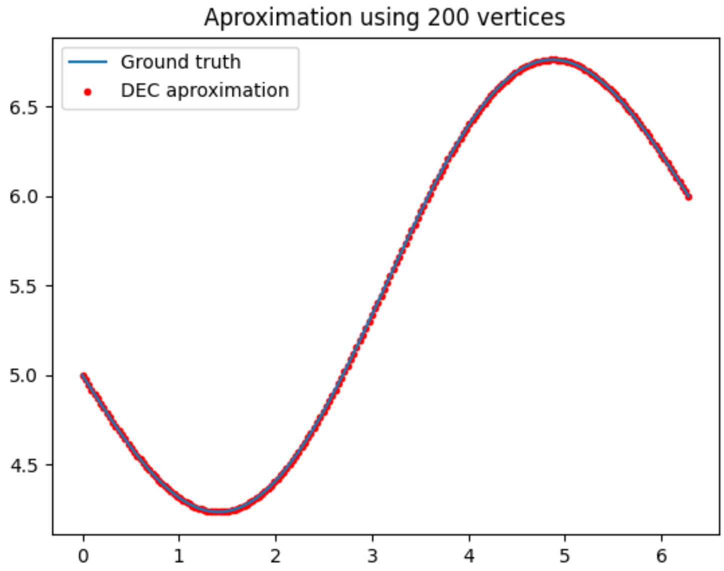


We get can compute the Mean Square Error:

$$MSE := \frac{1}{N}\sum_{i=1}^{N}(f_i - y_i)^2 = 0.021846378853579356.$$

where y_i is the ground truth for vertex i.

If we do the same but this time we discretize $[0, 2\pi]$ with a partition with 200 vertices we obtain a very accurate approximation



Such approximation has a very small error:

$$MSE = 3.4338479197291504 \times 10^{-9}.$$

3.2 Poisson equation in $\mathbb{R}^2$

This example was done for a project in a Discrete Differential Geometry course. It was implemented in Python. We consider Poisson equation with Dirichlet conditions

$$\Delta f = g, \quad (x,y) \in \Omega$$
$$f(x,y) = \varphi(x,y) \quad (x,y) \in \partial\Omega.$$

Most of the theoretical work has been done in Section 2. Moreover, this example is almost the same as the 1-dimensional case but this time we have more than 2 vertices in $\partial\Omega$. We first need to construct a mesh for domain Ω as the one shown in Figure 3.1 but a more refined one. Notice that there are many points of the discretization that are in $\partial\Omega$.

As explained before we only need to construct matrix

$$\Delta = -\mathrm{M}_0^{-1}\partial_1 \mathrm{M}_1 d_0$$

that depends only on the mesh. In this case matrices M_1 and M_0^{-1} are those shown in Section 2.3.2. Notice that Δ is a $N \times N$ matrix where N is the number of vertices in the mesh.

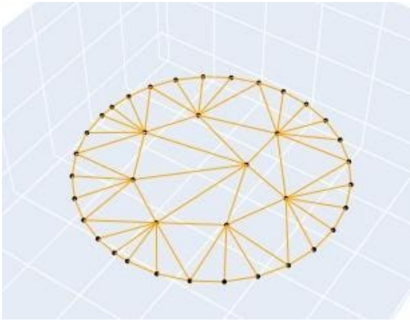


Figure 3.1: Discretization of Ω.

On the other hand, let

$$b = (g(v_1), g(v_2), \dots, g(v_N)) \in \mathbb{R}^N.$$

Then, we have the linear system

$$\Delta f = b.$$

However, notice that $f \in \mathbb{R}^N$ is such that $f_i = \varphi(v_i)$ if the ith vertex is in $\partial\Omega$. We therefore need to solve an auxiliar system only for the vertices in Ω:

$$\Delta_\Omega f_\Omega = b'$$

Here, as in the 1-dimensional case, Δ_Ω is the square matrix obtained by deleting rows and columns corresponding to vertices in $\partial\Omega$. In general, for a vector $u \in \mathbb{R}^N$ we denote by u_Ω the vector obtained by deleting entries corresponding to vertices in $\partial\Omega$. Then b' is given by

$$b' = b_\Omega - (\Delta h)_\Omega$$

where

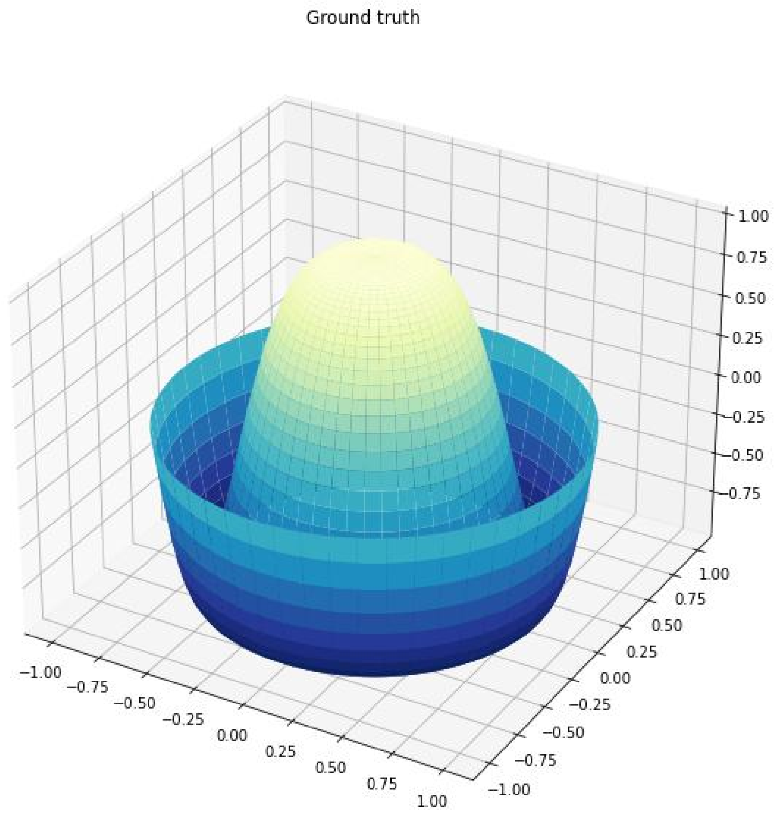
$$h_i = \begin{cases} f_i & \text{if} \quad v_i \in \partial\Omega, \\ 0 & \quad \text{otherwise.} \end{cases}$$

In this way, solving the system $\Delta_\Omega f_\Omega = b'$ we obtain $f(v_i)$ for vertices in Ω and we would have gotten an estimation of f for each vertex in the primal mesh.

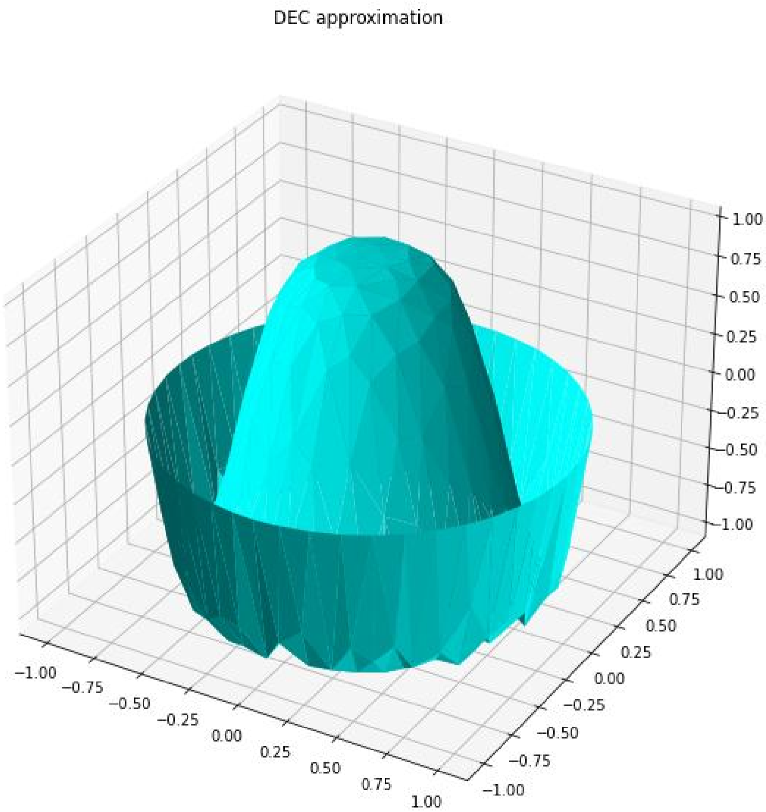
If we set

$$g(x, y) = -3\pi \left(2\pi \sin\left(\frac{3\pi}{2}(x^2 + y^2)\right) + 3\pi(x^2 + y^2)\cos\left(\frac{3\pi}{2}(x^2 + y^2)\right)\right)$$

and boundary condition $\varphi(x, y) = 0$. We get solution



On the other hand, DEC approximation is shown below:



3.3 Heat equation in one dimension

Here we aim to solve numerically Heat equation with Dirichlet boundary condition

$$\partial_t u = \partial_x^2 u, \quad x \in (a, b),$$
$$u(a,t) = \alpha, \quad u(b,t) = \beta.$$

and initial condition

$$u(x, 0) = \varphi(x).$$

Supposing that the primal mesh of $[a, b]$ has N vertices, let's think u as a function $\mathrm{u} : \mathbb{R} \to \mathbb{R}^N$ where ith entry $\mathrm{u}_i(t)$ corresponds to $u(v_i, t)$. We construct matrix Δ as in Section 3.1. However we need to discretize temporal derivative $\partial_t u$. We may try to use a partition $\{t_1, t_2, \ldots, t_M\}$ such that $t_{j+1} - t_j = h$, initial condition and an ODE numerical method as Euler's one:

$$\mathrm{u}(t_{j+1}) = \mathrm{u}(t_j) + h\Delta\mathrm{u}(t_j).$$

This scheme is called forward Euler [7] (Section 6.6). As mentioned in [7], forward Euler is not numerically stable. In addition we could not plug Dirichlet boundary conditions so easily.

Fortunately we could use another scheme. If we think backwards we can write:

$$\mathrm{u}(t_j) = \mathrm{u}(t_{j+1}) - h\Delta\mathrm{u}(t_{j+1}) = (I - h\Delta)\mathrm{u}(t_{j+1}).$$

This scheme is called backward Euler and it is far more stable [7]. Nevertheless it is computationally more expensive as we need to solve system for $u(t_j)$:

$$\mathrm{u}(t_j) = A\mathrm{u}(t_{j+1}), \quad A := I - h\Delta.$$

We can solve this system as we did in Section 3.1 plugging boundary conditions.

An example taken from [10] which was solve as a homework of the EDP course [1]

Problem 1. A rod has lenght $l = 1$ and (diffusion) constant $k = 1$. Its temperature satisfies the heat equation. Its left end is held at temperature 0, its right end ar temperature 1. Initially (at $t = 0$) the temperature is given by

$$\varphi(x) = \begin{cases} 5x/2 & , \quad x \in [0, 2/3] \\ 3 - 2x & , \quad x \in [2/3, 1] \end{cases}$$

We have PDE:

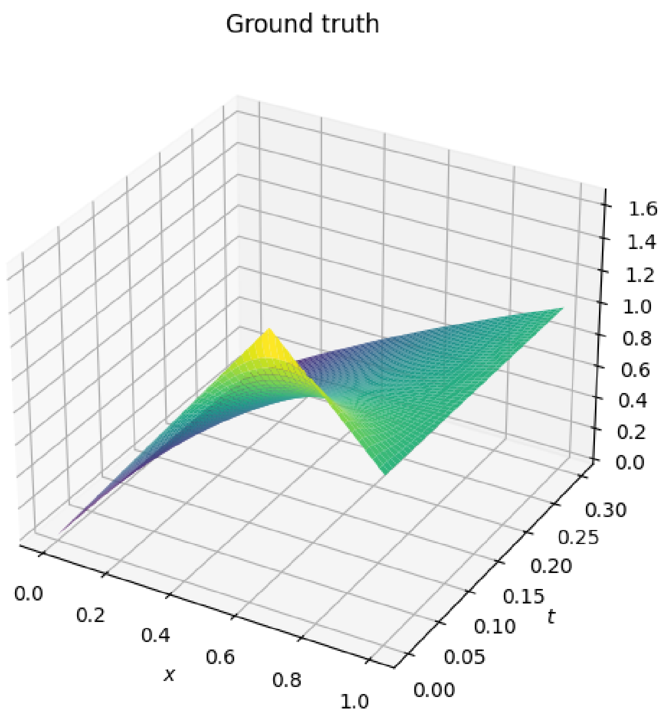
$$\begin{aligned} \partial_t u &= \partial_x^2 u, \quad x \in (0, 1), \\ u(0, t) &= 0, \quad u(1, t) = 1. \end{aligned}$$

and initial condition

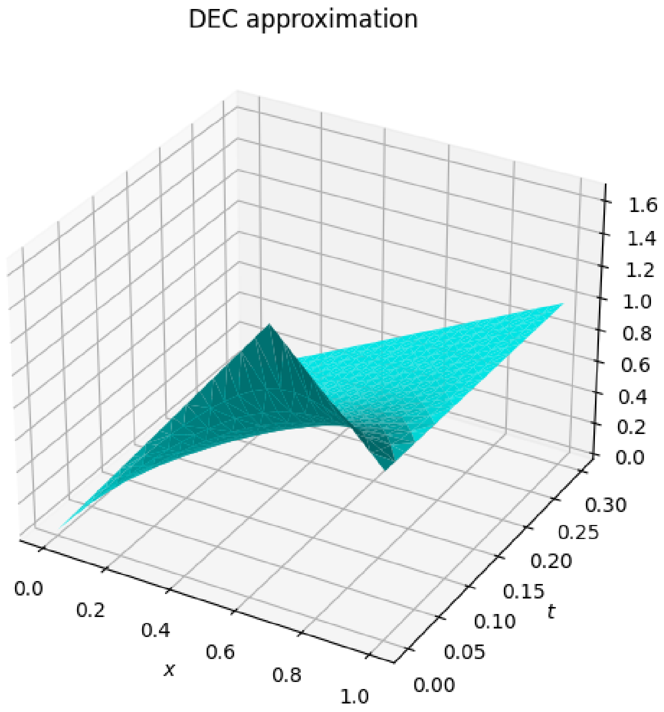
$$u(x, 0) = \varphi(x).$$

We saw that exact solution is given in terms of the serie:

$$u(x, t) = x + \frac{9}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp(-n^2\pi^2 t) \sin\left(\frac{2\pi n}{3}\right) \sin(n\pi x).$$



Here, ground truth is taken as the sum of first 100 terms of the serie. Now, we plot Discrete Exterior Calculus approximation


DEC approximation
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0.30
0.25
0.20
0.15
0.10
0.05
0.00
t
0.0
0.2
0.4
0.6
0.8
1.0
x

4 Conclusions and future work

Discrete Exterior Calculus is a very useful tool to solve numerically many types of Partial Differential Equations. It is an alternative to classical Finite Element approach.

The main contributions of this project are:

- Explanations of basic theoretical DEC concepts in a simple way but without forgetting important theoretical aspects.
- Connect theoretical definitions from Algebraic Topology and Hirani's thesis [6] with more practical and applicative concepts from [7] and [3].
- Definitions are visualized through images and figures with a more pedagogical approach.
- Practical and simple examples of problems that can be solve though DEC.
- DEC applications in Section 3 were made from scratch, except for Example 3.2 that was part of a project in a Discrete Differential Geometry course.

Some work and ideas that can be worked as a future work are:

- Implementations of applications in Section 3 can be done in a faster programming language as Julia or C++. Organized libraries con be programmed in order to work in a more organized and easier way.
- Explore discretizations of other differential operators different from the Laplace operator. For instance, in [5] and [4] an scheme to solve Darcy Flow Equation is presented.
- Explore other domains as surfaces in $\mathbb{R}^3$ or curves in the plane or space.
- Explore other boundary conditions other than Dirichlet's.

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