①

3331 E1
Sp 25
Sanders

1 classify

a) $x^2 \frac{du}{dx} + u^2 + x^2 = 0$

Not Linear

$\frac{du}{dx} = -\frac{(u^2+x^2)}{x^2}$ clearly NOT Separable

$\frac{du}{dx} = -\left((u/x)^2+1\right)$ IS Homogeneous

$A_u = x^2 \qquad A_x = u^2+x^2$

$A_{ux} = 2x \neq A_{xu} = 2u$ NOT Exact.

b) $x^2 \frac{du}{dx} + 2xu + x^2 = 0$

Linear

This is Linear

$\frac{du}{dx} = -\frac{(2xu+x^2)}{x^2}$ NOT Separable

$\frac{du}{dx} = -(2(u/x)+1)$ is Homog

$A_u = x^2 \qquad A_x = 2xu+x^2$

$A_{ux} = 2x = A_{xu} = 2x$ IS EXACT

②

c) $y \frac{dy}{dx} + x = 0$

$\boxed{\text{NOT LINEAR}}$

$y\,dy = -x\,dx$ $\boxed{\text{is separable}}$

$\frac{dy}{dx} = -\frac{x}{y}$ $\boxed{\text{is Homog}}$

$A_y = y$ $\quad A_x = x$ $\boxed{\text{IS EXACT}}$

$A_{yx} = 0$ $\quad A_{xy} = 0$

d) $e^y \frac{dy}{dx} + e^x y = 0$

$\boxed{\text{NOT Linear}}$

$\frac{e^y\,dy}{y} = -e^x\,dx$ $\boxed{\text{is separable}}$

$\frac{dy}{dx} = -\frac{e^x y}{e^y}$ $\boxed{\text{NOT Homog}}$

$A_y = e^y$ $\quad A_x = e^x y$ $\boxed{\text{NOT EXACT}}$

$A_{yx} = 0 \neq A_{xy} = e^x$

③

2a) Solve

$$\boxed{\frac{du}{dx} + \frac{1}{x} u = x^2}$$

First order linear
assume $x > 0$

$$\| $$

$$e^{-\log x} \frac{d}{dx}\left(e^{\log(x)} u\right) = x^2$$

$$\frac{1}{x} \frac{d}{dx}(xu) = x^2 \Rightarrow \int \frac{d}{dx}(xy) = \int x^3$$

$$xu = c + x^4/4 \qquad \boxed{u = \frac{c}{x} + \frac{x^3}{4}}$$

b) $$e^x \frac{du}{dx} = e^u \qquad e^{-u} du = e^{-x} dx$$

Separable

redefine $c = -c$

$$+e^{-u} = -c + e^{-x}$$

$$-u = \log(c + e^{-x})$$

$$\boxed{u = -\log(c + e^{-x})}$$

(4)

3.(1) $(u+x)\frac{du}{dx} + u+3x = 0$

This is homog, i.e. $\frac{du}{dx} = -\frac{u+3x}{u+x} = -\frac{\frac{u}{x}+3}{\frac{u}{x}+1}$

But let me see if it's exact (easier)

$\Lambda_u = u+x \qquad \Lambda_x = u+3x$

$\Lambda_{ux} = 1 \quad = \Lambda_{xu} = 1$ Yay!

I'll use the exact class.

$\Lambda_u = u+x$

$\Rightarrow \Lambda = \frac{u^2}{2} + xu + h(x)$

$\Lambda_x = u + h'(x) = u+3x \Rightarrow h(x) = \frac{3}{2}x^2$

$\Lambda = \frac{u^2}{2} + xu + \frac{3}{2}x^2 = \text{const},$

This is implicit, But use quad formula

$u^2 + 2xu + (3x^2 - c) = 0$ redefined c again.

$u = \frac{-2x \pm \sqrt{(2x)^2 - 4(3x^2 - c)}}{2}$

$$\boxed{u = -x \pm \sqrt{c - 2x^2}}$$

②

b) $xu \dfrac{du}{dx} + x^2 - u^2 = 0$

Not linear, Not sep.

But it is homog

Try exact

$A_u = xu \qquad A_x = x^2 - u^2$

$A_{ux} = u \neq A_{xu} = -2u$

$$\frac{du}{dx} = \frac{u^2 - x^2}{xu} = \frac{(u/x)^2 - 1}{u/x} \checkmark$$

$$\frac{u}{x} = v \qquad \frac{du}{dx} = x\frac{dv}{dx} + v$$

yay!

$$x\frac{dv}{dx} = \frac{v^2 - 1}{v} - v = \frac{v^2 - 1 - v^2}{v} = -\frac{1}{v}$$

$$\int v\,dv = \int -\frac{dx}{x} \Rightarrow \frac{v^2}{2} = c - \log(x)$$

assume $x > 0$

$$v = \pm\sqrt{c - 2\log x} \Leftarrow v^2 = c - 2\log(x)$$

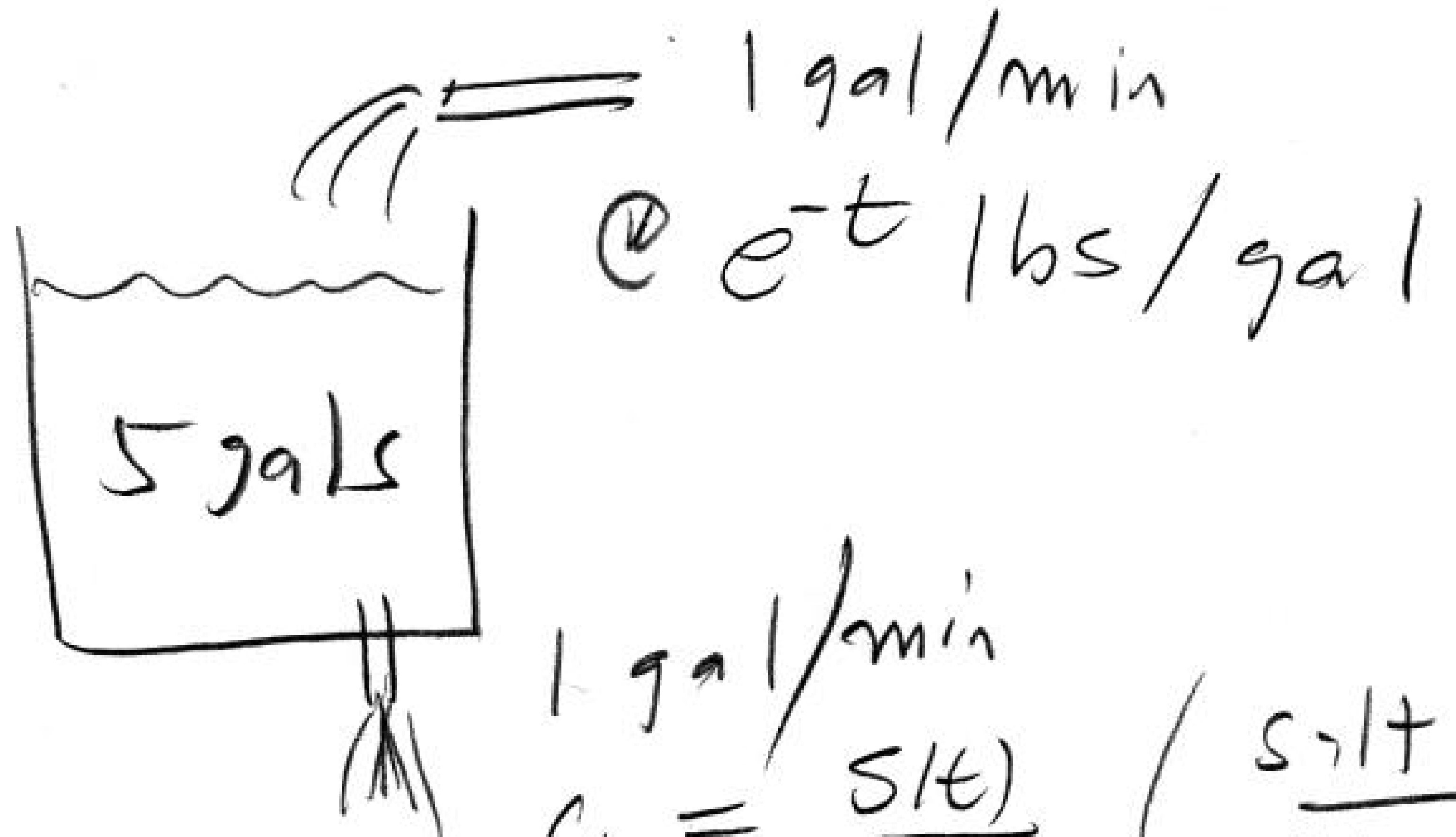
redefine

$$= \frac{u}{x}$$

$$\Rightarrow \boxed{u = \pm x\sqrt{c - 2\log(x)}}$$

⑥

#4)



$c = \frac{S(t)}{5}$ $\left(\frac{\text{salt in lbs}}{\text{gals in tank}}\right)$

$$\frac{dS}{dt} = \underbrace{\left(1\,\frac{\text{gal}}{\text{min}}\; e^{-t}\,\frac{\text{lbs}}{\text{gal}}\right)}_{\text{rate in}} - \left(1\,\frac{\text{gal}}{\text{min}} \cdot \frac{S}{5}\,\frac{\text{lbs}}{\text{gal}}\right)$$

a) So $\boxed{\frac{dS}{dt} = e^{-t} - \frac{S}{5} \text{ and } S(0)=0}$ because water is initially fresh

b) Solve $\frac{dS}{dt} + \frac{1}{5}S = e^{-t}$

Not separable but is 1st order linear

$$e^{-t/5}\frac{d}{dt}\left(e^{t/5}S\right) = e^{-t}$$

$$\int \frac{d}{dt}\left(e^{t/5}S\right) = \int e^{\left(\frac{1}{5}-1\right)t} = \frac{e^{\left(\frac{1}{5}-1\right)t}}{1/5-1}$$

$$\| $$

$$e^{t/5}S - c$$

$$\boxed{S = Ce^{-t/5} - \frac{e^{-t}}{4/5}}$$

$$\boxed{S(0)=0 \Rightarrow S(t) = \frac{5}{4}\left(e^{-t/5} - e^{-t}\right)}$$

(7)

5) Use the factorization to solve

a) $\left(\frac{d}{dx}+I\right)\underbrace{\left(\frac{du}{dx}\right)}_{\equiv v}=0$

$\Rightarrow \frac{dv}{dx}+v=0$

$\boxed{v=c_1e^{-t}}$ to get u

$\frac{du}{dx}=v=c_1e^{-t}$

$\boxed{u=c_2+c_1e^{-t}}$ redefine

b) $\left(\frac{d}{dx}+I\right)\underbrace{\left(\frac{du}{dx}+u\right)}_{\equiv v}=x$

$\frac{dv}{dx}+v=x$, $\int\frac{d}{dx}(e^xv)=\int\overbrace{xe^x}^{\text{byP}}$

$e^xv=c_1+((x-1)e^x)$

$v=c_1e^{-x}+(x-1)$

Now to get u

$\frac{du}{dx}+u=v=c_1e^{-x}+x-1$

(cont)

(8) Same trick

$$e^{-x}\frac{d}{dx}(e^{x}u) = c_1 e^{-x} + (x-1)$$

$$\int \frac{d}{dx}(e^{x}u) = \int c_1 + \int \overbrace{(x-1)e^{x}}^{\text{by Parts}}$$

$$e^{x}u = c_2 + c_1 x + (x-2)e^{x}$$

so $\boxed{u = c_2 e^{-x} + c_1 x e^{-x} + x - 2}$

$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u \leftarrow u = x-2$

$0 + 2(1) + (x-2) = x$ ✓