The Alexandrian Greek Period  

In 352 B.C., Philip II of Macedonians launched campaigns of conquest on Greece, it led to the destruction of classical Greece. Athens was defeated in 338 B.C.. Alexander the Great, son of Philip, took command in 338 B.C. to conquer Greece, Egypt, India and the cataracts of the Nile.

Figure 12.1  Alexandria and the Greco-Roman Museum

In 323 B.C., before he could complete his ambitious conquest, Alexander died. After his death, his generals began to fight each other for power. After several decades, the empire was finally split into three independent parts. The European portion became the Antigonid empire; the Asian part became the Seleucid empire; and Egypt became the third empire ruled by the Greek Ptolemy dynasty.
Gradually Greece and Macedonia fell under Roman domination and Greek mathematics was interrupted so that it became less and less important.

Most of the mathematical work generated in the Seleucid empire was basically continuation of Babylonian mathematics. After that period, the major creations of Greek mathematics occurred in the Hellenistic period, Ptolemaic empire, primarily in Alexandria.

![Figure 12.2](image)

Knowing the power and importance of culture, the kings of the Ptolemaic empire, who were wise Greeks, continued to pursue Alexander’s plan to build a great cultural center at Alexandria. These rulers respected the scholars from the great Greek schools such as those of Pythagoras, Plato and Aristotle, and brought them from all the existing centers of civilization to Alexandria using support from state funds.

About 290 B.C. Ptolemy Soter built a center for scholars, known as the Museum. ¹ Within the Museum there were poets, philosophers, philologists, astronomers, geographers, physicians, historians, artists, and mathematicians. The Museum was divided into four departments: literature, mathematics, astronomy, and medicine. Clearly mathematics occupied a dominant place within the Museum.

Nearby the Museum a library was built to store important documents and scholars’ work as well as to serve as a function for the general public to use. The library contained about

---

¹The word “museum” originated from the Greek mouseion which means house of the muses. Here the muses were the nine young goddess of classical times who watched over the welfare of astronomy, comedy, history, love poetry, music, oratory, tragedy, the epic and dance. The first museum set up by Ptolemy Soter in Alexandria was a centre of learning dedicated to the muses. So it was different from the public museums today.
750,000 volumes, including the personal library of Aristotle. Because of this, Alexandria became the center of the book-copying trade of the ancient world.

Mathematics in the Alexandrian Greek Period As we pointed out in Lecture 10, Euclid and Apollonius were Alexandrians; but Euclid organized the work that was previously done in the classical period. Remember that Apollonius also organized and extended classical Greek mathematics.

The other great Alexandria mathematicians are Archimedes (see below), Eratosthenes (see the footnote in Lecture 8), Hipparchus 2, Nicomedes 3, and Diophantus (see next Lecture).

Alexandrian geometry, following the practice of the Babylonians, was devoted to results which were useful in the calculation of length, area, and volume. Such results were scarce in Euclid’s *Elements*. The Alexandrians did not hesitate to use irrational numbers freely for lengths, areas, and volumes. The climax of this work was the development of trigonometry. There was also some progress in arithmetic, algebra, and mechanics.

Archimedes Archimedes, 287 B.C. - 212 B.C., was a Greek mathematician, physicist, engineer, inventor and astronomer. Eric Temple Bell, historian of mathematics, wrote: “Any list of the three ‘greatest’ mathematicians of all history would include the name of Archimedes. The other two usually associated with him are Newton and Gauss.” 4 Archimedes was the first to use mathematics to explain natural phenomena.

---

2Hipparchus, from 147 B.C. to 127 B.C., is considered the greatest ancient astronomical observer.

3Nicomedes was engaged in trying to solve the problems of doubling the cube and trisecting the angle, and made some contributions to these problems.

Archimedes is one of the few mathematicians whose life is known in detail thanks to the attention he received from classical authors such as Plutarch, Livy and Cicero. He was born in Syracuse (a Greek city in what is now Sicily) around 287 B.C. and did most of his important work there, though he may have studied for a time in Alexandria. He was the son of the astronomer Phidias.

Many of the stories about Archimedes relate to the ruler of Syracuse, King Hieron II, from which we know that Archimedes invented a number of his mechanisms for the benefit of Hieron: compound pulleys for moving ships, ballistic devices for the defense of Syracuse, and the model planetarium. The most famous story about Archimedes is the following. King Hieron possessed a crown in the form of a wreath which was to be placed on the statue of a god. Suspecting that the goldsmith might have replaced some of the gold given to him by an equal weight of silver, Heron asked Archimedes to determine whether the wreath was pure gold. Because the wreath was a holy object dedicated to the Gods, the wreath cannot be disturbed in any way. After puzzling over this for a long time, the solution came to Archimedes when he stepped into his bath. He realized that he would be able to test whether a crown was pure gold by weighing the crown against a mass of gold that would displace an equal amount of water. Upon this discovery he leapt from his bath tub with a shout of “Eureka!”

Before Archimedes there was no mathematical theory of mechanics at all, only the thoroughly incorrect mechanics of Aristotle. Archimedes’ most original contribution to science was his theoretical mechanics, including the law of the lever, centers of mass, equilibrium, and hydrostatic pressure. Archimedes was the first mathematician to derive quantitative results from the creation of mathematical models of physical problems.

---

[5] Plutarch, (46 - 120), was a Greek historian, biographer, essayist, and Middle Platonist known primarily for his books: *Parallel Lives and Moralia.*
**Method of Exhaustion**  The method of exhaustion is a method used to find the area of a shape by inscribing inside it a sequence of polygons whose areas converge to the area of the containing shape. If the sequence is properly constructed, the difference in area between the $n$th polygon and the containing shape will become arbitrarily small as $n$ becomes large. As this difference becomes arbitrarily small, the possible values for the area of the shape are systematically “exhausted” by the lower bound areas successively established by the sequence members.

![Figure 12.4](image-url)  Archimedes and method of exhaustion

The method of exhaustion was originated from some previous rigorous work by Eudoxus, but the theory was brought to full maturity by Archimedes. By his approach, Archimedes needed to “guess” these results first by non-rigorous methods, and proved them later by the method of exhaustion.

He was the first to apply it successfully to the parabola. He also found the areas of a variety of plane figures and the volumes of spaces bounded by all kinds of curved surfaces. These included the areas of the circle, segments of a parabola and of a spiral, and volumes of segment of cylinders, cones, and other figures generated by the revolution of parabolas, ellipses, and hyperbolas. In his tretise, *Quadrature of the Parabola*, Archimedes computed the area of a parabolic segment by summation of a geometric series. In fact, he gave the following:

$$a + \frac{1}{4}a + \left(\frac{1}{4}\right)^2 a + \ldots + \left(\frac{1}{4}\right)^n a + \frac{1}{3}\left(\frac{1}{4}\right)^n a = \frac{4}{3}a. \quad (1)$$

He first noticed $\left(\frac{1}{4}\right)^n a + \frac{1}{3}\left(\frac{1}{4}\right)^n a = \frac{1}{3}\left(\frac{1}{4}\right)^{n-1}$. Then he calculated
\[ a + \frac{1}{4}a + \left(\frac{1}{4}\right)^2a + \ldots + \left(\frac{1}{4}\right)^n a + \frac{1}{3} \left[ \frac{1}{4}a + \left(\frac{1}{4}\right)^2a + \ldots + \left(\frac{1}{4}\right)^n a \right] \]

\[ = a + \left(\frac{1}{4}a + \frac{1}{3} \cdot \frac{1}{4}a\right) + \ldots + \left(\frac{1}{4}\right)^n a + \frac{1}{3} \left(\frac{1}{4}\right)^n a \]

\[ = a + \frac{1}{3}a + \frac{1}{3} \cdot \frac{1}{4}a + \ldots + \frac{1}{3} \left(\frac{1}{4}\right)^{n-1} a \]

\[ = a + \frac{1}{3}a + \frac{1}{3} \left(\frac{1}{4}\right)a + \ldots + \left(\frac{1}{4}\right)^{n-1} a \].

This implies (1).

Beyond the success to find the area of parabola sections, Archimedes could not make it work in the case of two other famous curves: the ellipse and the hyperbola, which, together with the parabola, make up the family of conic sections. He could only guess correctly that the area of the entire ellipse is \(ab\pi\). In fact, these cases had to wait for the invention of integral calculus two thousand years later. We now know that the calculation of area of ellipse needs to calculate elliptical integrals and the calculation of area of hyperbola involves the concept of logarithm (see Lecture 22).

The method of exhaustion is seen as a precursor to the methods of calculus. The Greeks seemed to be very closed to the modern calculus, but due to lack of the concept of infinity and language of algebra, it is simply too far away to reach the goal. The development of analytical geometry and rigorous integral calculus in the 17th-19th centuries (in particular a rigorous definition of limit) subsumed the method of exhaustion so that it is no longer explicitly used to solve problems.

Figure 12.5  Archimedes. G.W. Leibniz said:
“He who understands Archimedes and Apollonius will admire less the achievements of the foremost men of later times.”
Calculation of $\pi$  Archimedes used the method of exhaustion as a way to calculate $\pi$ by filling the circle with a polygon of a greater and greater number of sides. The quotient formed by the area of this polygon divided by the square of the circle radius can be made arbitrarily close to the actual value of $\pi$ as the number of polygon sides becomes large.

In his brief treatise, Measurement of the Circle, Archimedes gave, in the Proposition 3,

$$\frac{31}{7} < \pi < \frac{310}{71},$$

i.e.,

$$3.140845 < \pi < 3.142857.$$

Other achievements  Archimedes also made many contributions in science. At a time when no method existed to describe very large numbers, in his Sand-Reckoner, he invented a system that allowed him not only to write down, but also to manipulate numbers of any magnitude (for example, $10^8$ in today’s notation). He found the famous spiral of Archimedes, which is given by $r = a\theta$ in modern polar coordinates and is still used in our calculus textbooks. The area of the spiral of Archimedes was also calculated by him. He discovered the famous postulate of Archimedes. His treatment on the above geometric problems led him to confront the solution of cubic equations. In physics, Archimedes discovered the laws governing floating bodies, thus establishing the science of hydrostatics. Also he calculated the centers of gravity of many solids and formulated the mechanical laws of levers. In astronomy, he performed observations to determine the length of the years and the distance to the planets.

In 1906, by accident, a Greek text of another work by Archimedes was discovered in the library of a monastery in Constantinople by a Danish philologist Johan Ludvig Heiberg. Sometimes between the 12th and 14th centuries, monks washed off the earlier text to provide space for a collection of prayers and liturgies. Fortunately, most of the contests still could be read with a magnifying glass. The manuscript contained many treaties of Archimedes including a only surviving copy of a largely unknown work entitled The Method.

Archimedes’ death  The story of Archimedes’ death has varying details. He was killed by a Roman soldier when Syracuse fell to the Romans under Marcellus in 212 B.C. According to Plutarch, 

---

6Of two unequal line segments, some finite multiple of the short one will excess the longer.

7A common practice caused by high costs of parchment.

“... as fate would have it, he was intent on working out some problem with a diagram and, having fixed his mind and his eyes alike on his investigation, he never noticed the incursion of the Romans nor the capture of the city. And when a soldier came up to him suddenly and bade him follow to Marcellus\(^9\), he refused to do so until he had worked out his problem to a demonstration: whereat the soldier was so enraged that he drew his sword and slew him.”

![Figure 12.6  Archimedes’ death](image)

Archimedes asked that his gravestone be inscribed with a figure and description of his favorite result, the relation between the volumes of the sphere and the cylinder.

\(^9\)Marcellus was a Roman general who gave explicit orders that Archimedes be spared.