The invention of logarithms by Napier is one of very few events in the history of mathematics — there seemed to be no visible developments which foreshadowed its creation. Its progress completely revolutionized arithmetic calculations.

**Background of logarithm** The idea of the logarithm probably could be originated from certain trigonometric formulas which transformed multiplication into addition or subtraction. For example,

\[ 2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta). \]

which was used by the 16th century astronomers.

Around the time of the 16th century, trigonometry functions such as sine and cosine were generally calculated to 7 or 8 digits, and these calculations were long so that occurring of errors was invertible. In order to decrease computational errors, astronomers realized that it would be greatly reduce the number of errors if the multiplication and divisions could be replaced by additions and subtractions. The above trigonometric identity was an example.
Sometimes in the 17th century, two men came up with the idea of producing an extensive table that would allow one to multiply any desired numbers together by performing addition. It is the Scot John Napier (1550-1617) and the Swiss Jobst Bürgi (1552-1632), both of them were working independently and Napier published his work first. Also, Bürgi’s book was printed under a unfavourable circumstances: during the Thirty Years War, and only few copies of Bürgi’s work were saved so that it went virtually unnoticed.

**John Napier’s life**  
John Napier was a Scottish mathematician, physicist, astronomer and astrologer. He was the eighth Baron of Merchiston in Scotland. After studying at home, when he was 13 years old, he entered St. Andrews, the oldest of universities in Scotland in 1563. No record showed that Napier got a degree at St. Andrews. Most likely he might choose to study abroad, as was the custom among young noblemen.

In 1571 Napier was back in Scotland. Soon he married and became a country man. After his father’s death in 1603, Napier moved to Merchiston Castle where he spend the rest of his life.

Napier was a theological writer. His book, “A Plaine Discovery on the Whole Revelation on Saint John,” was widely read, passing through 21 editions in various languages. In the book, Napier attacked the Catholic Church and argued that the Pope in Rome was the anti-Christ. This work made him more famous in his day than did his mathematics. Napier himself considered this work his great contribution to humanity.

Napier had a reputation as an inventor. His major inventions was a device with a hydraulic screw and revolving axle for pumping water out of coal mines. Also, like Archimedes, Napier had ideas for various engines of warfare which in fact were ultimately realised in the form of machine gun, tank and submarine today.

According one legend, Napier had used a jet-black rooster to discover which of his servants was stealing from him. Napier asked each of these servants to go, one by one, into a darkened room and instructed them to stroke the bird’s back. He assured them that the rooster would identify the thief by crowing. Napier had dusted the rooster with soot but these servants did not know. Finally, the thief, unwilling to touch the rooster, was the only one whose hand had not touched with soot.

**John Napier’s Mathematics**  
For Napier, mathematics was only a hobby. He wrote that he often found it hard to find time for the necessary calculations between working on theology. His best work is his invention of logarithms: he introduced a form of rod which can be used to multiply two numbers in a mechanical way and it was the origin of logarithm
today. Napier also has some other mathematical contributions in spherical triangles and trigonometric functions.

In 1614 Napier’s logarithmic tables, *Mirifici Logarithmorum Canonis Descriptio*, was published, which contained only a brief introduction without proof. In 1619 his second book, *Rabdologia*, two years after his death, gave the details of the theory and explained how to construct the tables.

![Figure 22.2 Jon Napier’s logarithm tables.](image)

Long before Napier, Michael Stifel published *Arithmetica Integra* in Nuremberg in 1544, which contains a table of integers and powers of $q$ that has been considered an early version of a logarithmic table, which is in terms of modern notation: for a fixed positive integer $q$,  

$$
\begin{array}{cccccccc}
q & q^2 & q^3 & q^4 & \ldots & q^n & \ldots \\
1 & 2 & 3 & 4 & \ldots & n & \ldots
\end{array}
$$

The rules $q^m \cdot q^n = q^{m+n}$ and $\frac{q^m}{q^n} = q^{m-n}$ were known for positive exponents. If we want to find $AB$ where $A = q^m$ and $B = q^n$, then we just need to do addition $m + n$ and then get the result $q^{m+n}$. This relation is the key idea behind logarithms; but whereas Stifel had in mind only integral values of the exponent, Napier’s idea was to extend it to a continuous range of values and think about as fellow: If we could write any positive number as a power of some given fixed number, which is called a base later, then multiplication and division of numbers would be equivalent to addition and subtraction of their exponents. In short,

\[\text{See } e, \text{ the Story of a Number, by Eli Maor, Princeton University Press, Princeton, New Jersey, 1994.}\]
multiplication can be reduced to addition, which greatly reduces the drudgery of numerical computations.

While the above table can be used only to calculate multiplication $A \cdot B$ only for special $A = q^m$ and $B = q^n$, Napier wants to make a table to calculate all sines and cosines for every minute between $0^\circ$ and $90^\circ$. At that time, the sine was not defined as a ratio, but as the length of a semi-chord in a circle of a certain radius. For Napier, $sine \ x = 10^7 \ sin \ x$ where $sin \ x$ is the modern sine and the sine of $90^\circ$ is $10^7$. Napier considers

\[
Cq^1 \ \ Cq^2 \ \ Cq^3 \ \ Cq^4 \ \ ... \ \ Cq^n \ \ ...
\]

How to make a such table such that $Cq^1, Cq^2, ..., Cq^n, ...$ cover all numbers from $sine \ 0^\circ$ to $sine \ 90^\circ$? Napier spent several years to decide to choose $C = 10^7$ and $q = 1 - 10^{-7}$.

Based on this idea with some more sophisticated technical treatment \(^2\), Napier spent 20 years, from 1594-1614, to finish his table of logarithm. Napier first named the exponent of each power its “artificial number” but later he decided on the term logarithm, which means “ratio number.”

**Significance of logarithm** Napier’s work was greeted with great enthusiasm, and the major application of logarithm is contributed to the advance of science, in particular to astronomy by making some difficult calculations possible. Kepler(1571-1630), who discovered the three laws of planetary motion, was an enthusiastic user of the newly invented tool, because it speeded up many of his calculations. In 1620, Kepler dedicated one of his publications to Napier, in which he gave a clear and impeccable explanation on this theory. Early resistance to the use of logarithms was muted by Kepler’s successful work.

Henry Briggs (1561-1631), an English mathematician in Gresham College in London, wrote: “I never saw a book that pleased me better, or made me wonder more.”

Friedrich Gauss had been able to predict the trajectory of the dwarf planet Ceres by surprising accurate mathematics. Reputedly when asked how he can do that, Gauss replied, “I used logarithms.”

The eighteenth century French mathematician Pierre-Simon de Laplace wrote that the invention of logarithms, “by shorting the labors, doubled the life of the astronomer.”

**Logarithm with base 10** Immediately after Napier’s invention, Henry Briggs in 1617 visited Napier. Both decided to improve table of logarithms by putting the base $b = 10$.

Briggs first table contained the common logarithms of all integers in the range 11000, with a precision of 8 digits. Since then, it has been used for the next 300 years. Initially, it might seem that since the common numbering system is base 10, this base would be more “natural” than base $e$. But mathematically, the number 10 is not particularly significant. The most “natural” number turns out to be $e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$ discovered by Euler.

The exponential function  From Calculus, we now know that the logarithm function $y = \ln x$ is the inverse function of the exponential function $y = e^x$. Concerning the exponential function, it reminds us a famous legend about the inventor of the game of chess: After inventing the game of chess, the King was so happy and asked the inventor what reward he would wish for. The man humbly requested that one grain of wheat be put on the first square of the chessboard, two grains on the second square, four grains on the third, and so on until all sixty-four squares were covered. Surprised by the modesty of this request, the king immediately ordered a sack of grain brought in, and his servants began to place the grains on the board. Beyond most people’s intuitive expectation, it became clear soon that the total number of grains is too big: it equals $18,446,744,073,709,551,615$, i.e., $1 + 2 + 2^2 + 2^3 + \ldots + 2^{63}$. If we placed that many grains in an unbroken line, the line would be some two light-years long!

Logarithm function and the area of the graph of hyperbola section  From the differential formulas $(e^x)' = e^x$ and $(\ln x)' = \frac{1}{x}$, we know that

$$\ln x = \int_1^x \frac{1}{t} dt,$$

which is the area of the graph of the function $f(t) = \frac{1}{t}$ for $t$ between 1 and $x$. 

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The curve \( y = \frac{1}{x} \) is equivalent to the hyperbola \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \), up to a rotation and a translation. Remember in the lecture 12 (p.78) that Archimedes could not find the area of the graph sections of ellipse and hyperbola.

Who was the first in history to find the relation between logarithm function and the area of the graph section of hyperbola? It was Gregoire (or Gregorius) de Saint-Vincent (1584-1667). By the modern language, he consider the graph of the function \( y = f(x) = \frac{1}{x} \). Fixing numbers \( a > 0 \) and \( 0 < r < 1 \). Consider a sequence of points \( a_1 = ar, a_2 = ar^2, \ldots, a_n = ar^n, \ldots \). Then \( 0 < \ldots < a_n < a_{n-1} < \ldots < a_3 < a_2 < a_1 < a \). Using the points \( a_1, a_2, \ldots \) to divide the graph of \( f(x) \) into many small sections with the vertices \( a_{n-1}, a_n, f(a_{n-1}) \) and \( f(a_n) \) for each \( n \). Then it associates many small rectangles \( A_1, A_2, \ldots, A_n \) where \( A_n \) has the height \( f(a_{n-1}) = \frac{1}{ar^{n-1}} \) and with the length \( a_{n-1} - a_n = ar^{n-1} - ar^n \) for each \( n \). Then the area of each \( A_n \) is \( (ar^{n-1} - ar^n) \cdot \frac{1}{ar^{n-1}} = 1 - r \). It found that as the distance from 0 grows geometrically, the corresponding areas grow in equal arithmetically. This remains true even when we go to the limit as \( r \) goes to 1. Although Vincent did not mention or did not realize logarithm \(^3\), this in turn implies that the relation between area and distance is logarithmic.

One of Saint-Vincent’s students, Alfonso Anton de Sarasa (1618-1667), wrote down this relation explicitly, marking one of the first times that use was made of the logarithmic function, whereas until then logarithms were regarded mainly as a computational device. Thus the quadrature (a historical mathematical term which means determining area) of the hyperbola was finally accomplished. \(^4\) Now we can understand why Archimedes was unable to find the area of sector of hyperbola: it involves logarithm!

**Newton’s formula** Newton found the remarkable series

\[
\text{Log}(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots
\]

This series converges for all values of \( t \) in the interval \(-1 < t \leq 1\) and in theory could be used to compute the logarithms of various numbers. Although Newton found it first, he did not publish it until Nicolaus Mercator (1620-1687) published in 1668 in his work entitled “Logarithmotechnia”. When learned of Mercator’s publication, Newton was bitterly disappointed and felt that his credit had been deprived. However, Newton continued his style: always confide his work only to a close circle of friends and colleagues. \(^5\)

**Tables of logarithms and slide rules** The invention of logarithms developed tables of logarithm and the portable version —– the slide rules.


A generation ago, the use of tables of logarithms was an integral part of secondary education. Moreover, the tables were either to base 10, earlier called “Briggsian” or to base $e$ earlier called “Napierian” or “hyperbolic.” The tables of logarithms with different precision were produced. In 1964, a table of logs to 110 decimal places was published under the auspices of the Royal Society.  

William Oughtred and others developed the slide rule in the 1600s based on the work by John Napier. The slide rule is used primarily for multiplication and division, and also for functions such as roots, logarithms and trigonometry, but is not normally used for addition or subtraction.

Before the advent of the pocket calculator, it was the most commonly used calculation tool in science and engineering. The use of slide rules continued to grow through the 1950s and 1960s. During the time, every engineer or engineering student had to have one slide rule. Our last century could not have been built without the slide rules.

![Slide rules with logarithmic scale.](image)

**Figure 22.4** Slide rules with logarithmic scale.

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