Lecture 31. Lagrange and Laplace

You may know Lagrange mean value theorem, Lagrange error estimate, Lagrange multipliers, Laplace equation, Laplace transform, ......

Joseph Lagrange  Lagrange, Joseph (1736-1813) is a French-Italian mathematician and mathematical physicist, who lived most of his life in Prussia and France, and was making significant contributions to all fields of analysis, to number theory, and to classical and celestial mechanics. He was the greatest mathematician of the eighteenth century. Lagrange succeeded Euler as the director of the Berlin Academy.

Lagrange was born of French and Italian descent (a paternal great grandfather was a French army officer who then moved to Turin). Before Joseph grew up, his father had lost most of his property so that young Lagrange had to rely for his position on his own abilities.
He was educated at the college of Turin. At beginning, Lagrange was not interested in mathematics but something else. At his age seventeen, by an occasional chance, he read a memoir by the English astronomer Edmond Halley (for whom Halley’s comet is named) that discussed the use of algebra in optics. Then he was excited, and he threw himself into mathematical studies. At the end of a year’s hard working, he taught himself mathematics to become an accomplished mathematician, and was made a lecturer in the artillery school.

In 1754 when he was 19 years old, Lagrange wrote several letters to Euler in which he described his result on solving the isoperimetrical problem \(^1\) which was conjectured for more than half a century. His method is about maximizing and minimizing functionals in a way similar to finding extrema of functions. Euler was very impressed with Lagrange’s results and also has some earlier result. Lagrange considerably simplified Euler’s earlier analysis which lead to the Euler - Lagrange equation of variational calculus. He became one of the founders of calculus of variations. This work at once placed Lagrange in the front rank of mathematicians then living.

![Royal Artillery School in Turin (now Turin’s Military Academy)](image)

**Figure 31.2** Royal Artillery School in Turin (now Turin’s Military Academy)

In 1755 he was made professor at the Royal Artillery School in Turin. In 1756, he was offered a superior position in Prussia, but he was too reluctant to leave home to accept it.

In 1760s, he was known as one of the greatest living mathematicians. In 1764 he received a prize from the Paris Academy of Sciences for an essay he wrote about the apparent oscillation of the moon that makes its features seem to appear and disappear.

\(^1\)The problem can be stated as follows: Among all closed curves in the plane of fixed perimeter, which curve (if any) maximizes the area of its enclosed region? This question can be shown to be equivalent to the following problem: Among all closed curves in the plane enclosing a fixed area, which curve (if any) minimizes the perimeter?
In 1766, after Euler left, Euler and Jean d’Alembert\(^2\) recommended Lagrange to take Euler’s place at Frederick the Great’s Berlin Academy. Next year, Lagrange married his cousin Vittoria Conti. Lagrange stayed in Berlin until 1787.

In Berlin, he published a large number of papers on topics including astronomy, the stability of the solar system, mechanics, dynamics, fluid mechanics, probability, and the foundations of the calculus.

Unfortunately, as his wife Vittoria’s health worsened, Lagrange nursed her and he was heartbroken when she died in 1783. Lagrange was deeply depressed.

When Frederick II died, Lagrange moved to Paris in 1787 and joined the Academie des Sciences in Paris. Though always welcome at social and scientific gatherings, he was always politely detached, sympathetic but uninvolved, and managed to stay aloof from the political turmoil that soon overwhelmed the country. This may enabled him to survive the 1789 revolution, which indeed took the lives of his friends Condorcet and Lavoisier.

Figure 31.3 1789 French revolution

Meanwhile, Legrange still made lots mathematical contributions: he contributed to the theory of fluid flow and introduced the Lagrangian function important for aerodynamics.

The revolution did stir some activity in Lagrange. In 1790, he joined a committee on weights and measures. In 1797, he became a professor of mathematics at the new École Polytechnique. His lectures were published as the first textbooks on real analytic functions.

\(^2\)Jean le Rond d’Alembert (1717 - 1783) was a French mathematician, mechanician, physicist and philosopher. D’Alembert’s method for the wave equation is named after him. In calculus, there is ratio test under his name, a test to see if a series converges.
In 1792 Lagrange married R. Le Monnier. His interest in life and mathematics revived, and he even made some brilliant contributions to celestial mechanics. Lagrange was honored by Napoleon Bonaparte, who named him to the Legion of Honour and a count of the empire in 1808. On April 3, 1813, he was named grand croix of the Ordre Impérial de la Réunion. He died a week later and was buried in the Pantheon in Paris.

**Number theory**  In number theory, Fermat was essentially alone for the whole of the 17th century. Euler was also essentially alone until he was jointed by Lagrange. We list some of Lagrange’s work:

- Give the first proof of a theorem stated by J. Wilson that \( n \) is prime if and only if \( n \) divides \( (n-1)! + 1 \).
- Find all integer solutions of the Diophantine equation \( x^2 - ay^2 = 1 \).
- The solution of numerous problems posed by Fermat (for example, every prime number \( p \) of the form \( p = 8n + 1 \) could be written as \( p = a^2 + 2b^2 \) for suitable integers \( a \) and \( b \)).
- Prove that every positive integer can be expressed as the sum of four integer squares.

![Figure 31.4 Lagrange points](image)

**Lagrange points and black holes**  Lagrange believed that in a two-body system, such as the Earth and the Sun, there would be five points nearby where an object could be sent and remain in place. He theorized that at certain points, the gravity of two bodies would cancel out one another and halt the motion of the spacecraft, keeping it in one spot. Lagrange was right. And now, NASA is using those points in space as parking spots for spacecraft which are discovering the secrets of the universe.
Lagrange was one of the first scientists to postulate the existence of black holes and the notion of gravitational collapse.

**Lagrange’s algebraic foundation of analysis** English philosopher bishop George Berkeley’s criticism on calculus was well informed and efficient. To Berkeley’s criticism, C. Maclaurin and D’Alembert tried to give answer but not succeed.

The approach which most strongly influenced the future development of analysis was proposed by Lagrange. The fundamental concept of his approach was that a function where, completely in line with Euler’s algebraic view, a function was an analytic expression. Lagrange viewed all functions as power series and attempted to reduce all mechanics to the analysis of power series, without use of geometry. But this became intelligible only in the domain of complex analytic functions, which was emerged from the geometric theory of complex integration discovered by Gauss and Cauchy.

**Contribution to analytic mechanics** Based on works of Newton, Bernoulli brothers and Euler, Lagrange developed so-called “variational principles” to the problems of equilibrium and motion of material system. In the second half of 18th century, mainly thanks to Lagrange, such principles were established. Lagrange presented his results in two memoirs in 1761. According to him the essential achievement of his memoir was the reduction of the solution of the problem of motion for any system of bodies to a quite simple application of the quasi-algebraic principle of undetermined coefficients.

![Figure 31.5 Pierre-Simon Laplace](image)

**Laplace** Pierre-Simon Laplace (1749 - 1827) was a French mathematician who is the last of the leading eighteenth century mathematicians. He formulated Laplace’s equation, and
pioneered the Laplace transform which appears in many branches of mathematical physics, a field that he took a leading role in forming. The Laplacian differential operator, widely used in applied and pure mathematics, is also named after him. His work was also pivotal to the development of mathematical astronomy and statistics.

He entered the University of Caen at 16, intending to study theology, but soon decided that his true interest was in mathematics. He remained at Caen for five years where he wrote his first mathematical paper.

Then he went to Paris to seek a mathematical position. He submitted several recommendation letters to d’Alembert who was the leading French mathematician, but was not noticed. Then Laplace wrote d’Alembert a brilliant letter on the general principles of mechanics. This caught d’Ambert’s eye and it brought the enthusiastic replay, “You need no introduction; you have recommended yourself; my support is your due.” Mainly through d’Alembert’s influence, Leplace was appointed professor of mathematics at the Paris Ecole Militaire in 1769. Because his mathematical accomplishment, he won election to the Académie of Science in 1773.

Examine a student whose name was Napoleon  In 1785, Laplace happened to examine a student whose name was Napoleon Bonaparte and the examinee passed. When Napoleon took power in 1799, Laplace offered his complete support to his old friend. During revolution, Laplace served as a member of the Commission on Weights and Measures, but was eventually


\[^{3}\text{Napoleon Bonaparte (1769 - 1821), was a military and political leader of France and Emperor of the French as Napoleon I.}\]

211
dismissed for not being a strong republican. When Napoleon teasingly remarked to Laplace that God is not mentioned in his work the *Mechanique Celeste*, Laplace replied, “Sire, I did not need that hypothesis.” When Napoleon later reported this reply to Lagrange, the latter remarked, “Ah, but that is a fine hypothesis. It explains so many things.”

**One of the greatest science work** Laplace’s greatest achievement was the *Traité de Mécanique Céleste*, one of the great scientific works of the early 1800s, published in five large volumes over 26 years (1799-1825), which was to “solve the great mechanical problems of the solar system.” Within its 2000 pages, it contained all the important discoveries of Newton, Clairaut, d’Alembert, Euler and Lagrange and himself.

The book is concise. Whenever Laplace was satisfied that a conclusion was correct, he was content to insert the optimistic remark, “It is easy to see that ...” and give the result without further explanation. Astronomer N. Bowditch observed, who translated four of the five volumes into English, “I never came across one of Lapace’s ‘Thus it plainly appears’ without feeling sure that I had hours of hard work before me to fill up the chasm and find out and show how it plainly appears.”

![Figure 31.6](image)

**Laplace equation** Laplace developed the idea of the potential, which was appropriated from Lagrange who had used it in his memoirs. Laplace shewed that the potential always satisfies the differential equation

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \]

Laplace also made great contribution to the probability theory (Laplace considered probability theory to be simply “common sense reduced to calculus”) and mathematical astronomy (stability of the solar system).

Laplace, called the “Newton in France,” died in 1827 at age 78. When those around his deathbed were recalling to him the great discoveries he had made, Laplace replied. “What we know is but a little thing; what we are ignorant of is immense.”

Figure 31.7 Around the end of the 18th century

**Pessimism view at the end of the eighteenth century** It is a curious fact that toward the end of century some of the leading mathematicians expressed the feeling that the field of mathematics was somehow exhausted and the development of mathematics seemed to have reached its climax.

“Does it not seem to you that sublime geometry is tending to become a little decadent?” Wrote Lagrange to d’Alembert in 1772, “It has no other support than you and Mr. Euler.” In 1781, Lagrange wrote to d’Alembert. “It appears to me also that the mine [of mathematics] is already very deep and that unless one discovers new veins it will be necessary sooner or later to abandon it.” Euler and d’Alembert agreed with Lagrange that mathematics had almost exhausted its ideas and they saw no new great minds on the horizon. Argos later expressed a sentiment which may help us to understand this feeling:


6Francois Jean Dominique Arago (1786 - 1853) was a French Catalan mathematician, physicist, astronomer and politician
Five geometers — Clairaut, Euler, d’Almbert, Lagrange and Laplace — shared between them the world of which Newton had revealed the existence. They explored it in all directions, penetrated into regions believed inaccessible, pointed out countless phenomena in those regions with observation had not yet detected, and finally — and herein lies their imperishable glory — they brought within the domain of a single principle, a unique law, all that is most subtle and mysterious of the future; when the centuries unroll themselves they will scrupulously ratify the decision of science.”

Nevertheless, a new generation, inspired by the new perspectives opened by the French Revolution was to show how unfounded this pessimism was. This great new came in part form Gauss (see Lecture 32) in Göttingen, in part from Cauchy (see Lecture 34) in France, and also from many others.

![Figure 31.8](image-url)  
*Figure 31.8* Perspective view, copper-plate engraved print, hand-coloured, c1790. Tuileries Palace in Paris (doesn’t exist anymore) In this print, we can see the life along the Seine River in Paris, the costumes of parisiens, and in particular, the Tuileries Palace.