Lecture 35. Cauchy, Who Set the Foundation of Analysis

Figure 34.1 Cauchy was living in a cottage in Arcueil

Augustin Cauchy

Augustin Louis Cauchy (1789-1857) was a French mathematician and is one of the greatest modern mathematicians. Cauchy pioneered the fields of analysis, both real and complex, and brought precision and rigor to mathematics. His name is prominent in almost any analysis textbook. He also studied differential equations, determinants, probability, groups and mathematical physics. Cauchy has credit for 16 fundamental concepts and theorems in mathematics and mathematical physics, more than any other mathematician. He is also known as one of the most prolific writers in the history of science, and he wrote 789 papers, a quantity exceeded only by Euler and Cayley. His collected works were published in 27 volumes.

Cauchy was born in Paris in 1789, only a month after the storming of the Bastille. His father, a government official and a lawyer, recognized the coming revolution and quickly moved his family to a country cottage in Arcueil. Staying at Arcueil, the family was poor and life was hard. This early poverty caused Cauchy to remain in a state of ill-health for the rest of his life.

During his eleven years stay at the cottage, Augustin received a classical education from his father, who wrote his own textbooks, and received Catholic religious training from his
mother. This training would influence the rest of his life. Throughout his life Cauchy held extreme anti-revolutionary and pro-royalist views.

During this early period, he had the benefit of contact with the famous mathematician Lagrange who came to visit Pierre-Simon Laplace, a neighbor of Cauchy. Augustin’s talent was recognized by both great mathematicians. Both, after seeing the young boy’s work, encouraged him to continue in mathematics. As Lagrange once predicted, he would eventually outdo both of them, but advised his father not to show him a mathematics book before he was 17.

![Figure 34.2](image1)

Figure 34.2 Cauchy studied engineering at École Polytechnique.

As Napoleon took power at the end of the eighteenth century, the political situation stabilized, and the Cauchy family returned to Paris in 1800. Cauchy completed his study at secondary school in 1805, interested in a scientific career. He entered the École Polytechnique in 1805 with a major in engineering, and transferred to the Ecole des Ponts in 1807.

In 1810, Cauchy took a position as an engineer in Napoleon’s army at Cherbourg. He carried with him the two mathematical books by Laplace and Lagrange. During his busy schedule, he found time to study mathematics. During his three years there, he produced several significant mathematical papers. His first important mathematical work was the solution of a problem posted by Lagrange: to show that any convex polyhedron is rigid. Cauchy’s theorem partially settled a conjecture of Euler that any closed surface is rigid.

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1Polytechnique is established during the French Revolution in 1794, which is the most prestigious educational establishment in France.
In fact, Cauchy’s result is the best possible because Connelly (1977) found a non-convex polyhedron which is not rigid.

All this mathematical output also accomplished to ruin his health, for sake of his health, Cauchy was persuaded by Lagrange to abandon the profession of engineering and to devote himself to mathematics.

The famous Cauchy integral theorem was submitted to the French Academy in 1814, and carried him into the mathematical mainstream. With all his efforts focused on mathematics, Cauchy became a star on the mathematics scene.

Cauchy won a grand prize from the French Académie des Sciences in 1816 for a 300-page paper on waves at surface of a liquid. In the same year, Cauchy became a professor at the Ecole Polytechnique. At the age of 27, he was elected to the Academy of Sciences in Paris. Cauchy and Alorse de Bure were married in 1818 and had two daughters.

In Switzerland Cauchy became a professor at the University of Turin.

In 1830 after the overthrow of King Charles X, all members of the Academy were obligated to swear an oath of allegiance to the new king. Having already taken an oath to Charles, as a good royalist, Cauchy refused. This meant he had to resign his chair, but Cauchy went further: he left his family and followed the old king into exile in Switzerland. There he became a professor at the University of Turin. Two years later, Charles X, now in exile, asked Cauchy to supervise the education of his heir Henri. He agreed and was joined with his family in Vienna. His new duties overwhelmed him and his mathematical work slowed down.

He did not return to Paris until 1838. He still refused to take the oath and constantly struggled to find and hold a position. Finally in 1848, the oath was abolished and he resumed
his old posts. He returned to the Sorbonne and kept up a steady flow of mathematical papers. Augustin Cauchy died on May 23, 1857, after contracting a fever on a trip to the country to help restore his health. His last words were, “Men die but their works endure.”

Figure 34.4 Cauchy’s textbook Cours D’Analyse

**Cauchy’s contribution to Real Analysis**  Like Euler, Cauchy’s work embraced almost all mathematical branches. Cauchy’s main contribution was setting the groundwork for rigor in analysis and all of mathematics.

Over the previous centuries, mathematicians had tried in vain to discover what were the underlying principles of calculus and many had asserted that Newton’s discovery was flawed. There was a crack in the foundations of Calculus. For example, from the geometric series

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\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \ldots,
\]

Leibniz had suggested that \(1 - 1 + 1 - \ldots = \frac{1}{2}\); Euler held that from \(\frac{1}{1+1} = \frac{1}{4}\) one could conclude that \(1 - 2 + 3 - 4 + 5 - \ldots = \frac{1}{4}\); Grandi referred to the paradoxical result \(1 - 1 + 1 - \ldots = 0 + 0 + \ldots + = \frac{1}{2}\). This, he suggested to Leibniz, could be compared with the mysteries of the Christian religion that an absolutely infinite force created something out of absolutely nothing.  

Cauchy took the first step toward unifying the science. He defined continuity and derivative in terms of the limit, and he gave the first good definition of the limit as:

“When the values successively assigned to the same variable indefinitely approach a fixed value, so as to end by differing from it as little as desired, this fixed value is called the limit of all the others.”

Cauchy gave a form of the \((\epsilon, \delta)\)-definition of limit, in the context of formally defining the derivative, in the 1820s. The precise \((\epsilon, \delta)\) - definition of limit was later formulated by Weierstrass:

\[ f(x) \text{ is continuous at } x_0 \text{ if } \forall \epsilon > 0, \exists \delta > 0 \text{ such that } |f(x) - f(x_0)| < \epsilon, \text{ whenever } |x - x_0| < \delta. \]

Cauchy systematized its study and gave nearly modern definitions of limit, continuity, and convergence, and developed the theory of functions, and differential equations. He was the first to prove Taylor’s theorem rigorously, establishing his well-known formula of the remainder. His work provided a basis for the calculus. Cauchy is especially famous for his work with convergent series. The well-known Cauchy criterion determines if an infinite series is convergent or divergent. “Cauchy sequence” is a basic concept.

While attending Cauchy’s lecture on convergence, Laplace became panicked and rushed home. Laplace had just finished his masterpiece that used infinite series as its backbone so that he had to desperately check each one for convergence. Fortunately, all of the infinite series in his books were convergent.

![Figure 34.5 Weierstrass](image)

**Cauchy’s wrong proof for uniform convergence**  Cauchy did not correctly distinguish between continuity at a point versus uniform continuity on an interval, due to insufficient

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3For Weierstrass, see the next lecture.
rigor. Notably, in his 1821 *Cours d'analyse*, Cauchy gave a famously incorrect proof that the (pointwise) limit of (pointwise) continuous functions was itself (pointwise) continuous. The correct statement should be that the uniform limit of uniformly continuous functions is uniformly continuous. This required the concept of uniform convergence, which was first observed by Weierstrass’s advisor, Christoph Gudermann, in an 1838 paper, where Gudermann noted the phenomenon but did not define it or elaborate on it. Weierstrass ⁴ saw the importance of the concept, and both formalized it and applied it widely throughout the foundations of calculus.

**Cauchy’s contribution to Complex Analysis** The most original creation of the nineteenth century was the theory of functions of a complex variable. ⁵ It is useful in many branches of mathematics, including number theory, algebra, topology, PDE, dynamical systems, fractal geometry, as well as in physics including hydrodynamics, thermodynamics, electrical engineering, and string theory⁶. Many mathematicians, Euler, Gauss, Cauchy, Weierstrass, Riemann and many more in the 20th century did pioneer work.

The most famous work by Cauchy in complex analysis were the Cauchy integral theorem and the Cauchy integral formula.

Cauchy integral theorem claims that for a holomorphic function $f$, the integral of $f$ along a path only depends on the initial point and the terminal point of the path. Gauss already mentioned this theorem in a letter to Bessel on 1811. He wrote: “This is a very beautiful theorem, whose not-so-difficult proof I will give when an appropriate occasion comes up.” ⁷ But Gauss did not publish it until 1831. Gauss’ result was unknown to Cauchy.

In 1825 Cauchy published a paper on integrals in a complex domain which might be considered his masterpiece, which was based on some of his early work in 1814. In this paper, Cauchy not only proved the theorem, but also focused on its applications. ⁸

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⁴Karl Theodor Wilhelm Weierstrass (1815 - 1897) was a German mathematician who is often cited as the “father of modern analysis.”


⁶String theory is a developing theory in particle physics which attempts to reconcile quantum mechanics and general relativity.


Cauchy’s other mathematical contributions  Cauchy is famous in the field of mathematics for two main reasons: his numerous contributions to the science and his immense publishing. His works spanned every branch of mathematics and are simply too long to list. Here are some of his works.

- He developed the theory of groups and substitutions, and proved that the order of any subgroup is a divisor of the order of the group.
- He contributed to the development of mathematical physics and, in particular, aeronautics.
- He proved Fermat’s three triangle theorem.
- He contributed significant research in mechanics, substituting the notion of the continuity of geometrical displacements for the principle of the continuity of matter.
  He published classical papers on wave propagation in liquids and elastic media.
  He substituted the concept of the continuity of geometrical displacements for the principle of the continuity of matter
- In optics, he developed wave theory, and his name is associated with the simple dispersion formula.
- He invented the name for the determinant and developed the theory of determinants.
- In astronomy he described the motion of the asteroid Pallas.
Cauchy is also famous for his writings. He overwhelmed the mathematics world with the number and size of his works. All in all, his total output included 789 full length papers. It was not uncommon for him to finish two such papers in one week. In addition, these works tended to be rather long, sometimes extending for over 300 pages. In fact, after submitting several large papers to be published in the weekly bulletin, the Academy was forced to limit submissions to four pages to save their small budget from Cauchy’s pen. However, all this writing did get his work out into the public and spread his ideas.