January 30, 2025[*]

The numbers of statements/pages/etc. in the textbook are marked with a †. E.g., Section 1.7† means that 1.7 is the same number as in the textbook, page 70† refers to page 70 of the textbook.

Contents

0 Terminology

Here are some of the terms, with a brief description

- in general z stands for a complex number, $z \in \mathbb{C} \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; $z = x + iy$ denotes its real and imaginary parts (so x, y are real numbers).
- for a complex-valued function we often denote $f = u+iv$ for its real and imaginary components
- $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk in $\mathbb{C}$.
- $f(z)$ is differentiable at z_0 means that f has a (complex) derivative at $z_0 \in \mathbb{C}$
- $f(z)$ is analytic at z_0 it is differentiable on a neighborhood of z_0
- $f : \mathbb{C} \to \mathbb{C}$ is an entire function if it analytic on all of $\mathbb{C}$
- a singular point (a.k.a. singularity) of f is a point where f is not analytic but is the limit of points where f is analytic; e.g. 0 is a singularity of $f(z) = 1/z$; see page 70†
- a singularity is removable if can redefine the function at that singular point to make it analytic; e.g. 1 for $(z^2 - 1)/(z - 1)$, which equals $z + 1$ except at $z = 1$.

[*] If you find typos/errors/omissions/etc. please let me know, will correct them for a subsequent edition.

1 Chapter 1†: Complex Numbers

1.7 Section 1.7†: The Riemann sphere and stereographic projection

Without proof: the stereographic projection preserves angles (i.e., is is *conformal*) and takes circles on the Riemann sphere to circles or lines in the complex plane (see Figure 1.23†).

2 Chapter 2†: Analytic Functions

2.2 Section 2.2†: Limits and Continuity

Note the definition of convergence to ∞ on page 62†: $z_n \to \infty \iff |z_n| \to \infty$, and similarly $\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} |f(z)| = \infty$. See problem 23†: convergence to infinity corresponds via the stereographic projection to convergence to the North pole.

2.3 Section 2.3†: Analyticity

Definition 2.1 (See e.g. top of page 70†)

f is *analytic* at a point z_0 if is differentiable on a neighborhood of z_0.

f is *entire* if it is analytic on the whole complex plane.

A *singular point (a.k.a. singularity)* is a point where f is not analytic but which is the limit of points where f is analytic.

E.g., the complex exponential is an entire function.

2.4 Section 2.4†: The Cauchy-Riemann Equations

"Cauchy-Riemann equations" is usually abbreviated CR. For $f = u + iv$ they are:

$$\begin{cases} \frac{\partial u}{\partial x} = & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = & -\frac{\partial v}{\partial x} \end{cases}$$

Remark 2.1 (How to remember the Cauchy-Riemann equations)

- Require that the derivatives of $u + iv$ in the horizontal and vertical directions (i.e, w.r.t x and iy) be equal (see top of page 74†).

- Another explanation:
  - Multiplication by a complex number $z = a + ib$ on $\mathbb{C}$ is a linear transformation $\xi \in \mathbb{C} \mapsto z\xi \in \mathbb{C}$. Seeing $\mathbb{C} = \mathbb{R}^2$, this linear transformation has a 2×2 matrix, which is (recall the linear algebra course)

    $$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{2.1}$$

    This gives a *representation* of $\mathbb{C}$ as 2×2 matrices with real entries, consistent with both addition and multiplication in $\mathbb{C}$ (that is, $\mathbb{C}$ is isomorphic to this set of 2×2 matrices)
  - CR at z_0 for $f(x + iy) = u + iv \iff$ its *Jacobian matrix* at z_0 (that is, the matrix of partial derivatives),

    $$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

    is of the form (2.1).

2.5 Section 2.5†: Harmonic functions

Harmonic functions are solution to the Laplace equation; they appear in many physical phenomena.

Theorem 2.2

- *If f is complex-analytic, then its real part (and therefore its imaginary part as well) are harmonic.*
- *Conversely, if u is harmonic on a simply connected domain*[1] *G then there is a function v on G such that $u + iv$ is complex-analytic.*

[1] That is, a domain that has “no holes”. E.g., $\mathbb{C}$ and $\mathbb{D}$ are simply connected, but $\mathbb{D}$ without the origin or an annulus are not simply connected.