MATH 2331 EXAM 2 REVIEW NAME______________
March 26, 2019

NO CALCULATORS!

1. Find the eigenvalues and eigenvectors of the following matrices. Determine whether each matrix is diagonalizable. For those that are diagonalizable, find P such that $P^{-1}AP$ is diagonal. 32 pts

a. $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ Upper triangular, $\lambda = 2, 2$. Eigenspace for $\lambda = 2$ is the span of $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. Note rank(A – 2I) is 1. A is not diagonalizable.

b. $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ Upper triangular, $\lambda = 2, 3$.

Eigenspace for $\lambda = 2$ is the span of $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ = Nul(A-2I).

Eigenspace for $\lambda = 3$ is the span of $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ = Nul(A-3I).

A is diagonalizable with $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

c. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. $\det(A - \lambda I) = (\lambda - 2)^2 - 1 = 0$ if $\lambda = 2 \pm 1 = 1, 3$.

Eigenspace for $\lambda = 1$ is Nul(A-I) = $Span\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

Eigenspace for $\lambda = 3$ is Nul(A-3I) = $Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

A is diagonalizable with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

12 pts

2. Let $B = \left\{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}\right\}$, and $\boldsymbol{x} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ 10 pts

Assume **x** is in the span of B. Find the coordinates of **x** with respect to the basis B.

Solve $c_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \boldsymbol{x} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$. Solution $c_1 = 2$, $c_2 = -2$. The coordinate vector with respect to B is $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

3. If $\boldsymbol{A}=\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$, $T\left(\begin{bmatrix}x\\ y\end{bmatrix}\right)=\mathbf{A}\begin{bmatrix}x\\ y\end{bmatrix}$, and $D=\{(x,y):\ 0\le x\le 1,\ 0\le y\le 1\}$:

a. Sketch T(D) and label the vertices. 4 pts

T(D) is a filled parallelogram with vertices (0,0), (2,1), (1,3), and (3,4).

b. What is the area of T(D)? 6 pts

Area = $|\det(A)|$ = 5.

4. Let $A=\begin{pmatrix}1 & 3 & 3 & 3 & 4\\ 2 & 6 & 5 & 7 & 10\\ -1 & -3 & -4 & -2 & -2\end{pmatrix}$. Find a basis for each of Nul(A), Col(A), Row(A), and Nul(A^T). 24 pts

First find the reduced echelon form for A. To find a basis for Nul(A^T), augment with an unknown column vector $\boldsymbol{b}\in\boldsymbol{R}^3$:

$[A|b]=\begin{bmatrix}1 & 3 & 3 & 3 & 4 & b_1\\ 2 & 6 & 5 & 7 & 10 & b_2\\ -1 & -3 & -4 & -2 & -2 & b_3\end{bmatrix}$. The reduced echelon form of this is:

$R=\begin{bmatrix}1 & 3 & 0 & 6 & 10 & -5b_1+3b_2\\ 0 & 0 & 1 & -1 & -2 & 2b_1-b_2\\ 0 & 0 & 0 & 0 & 0 & 3b_1-b_2+b_3\end{bmatrix}$. The expression $3b_1-b_2+b_3$ tells us that the third row of R is $3a_1-a_2+a_3$, where a_1,a_2,a_3 are the rows of [A|b]. This means that the vector $(3,-1,1)$ is orthogonal to each of the columns of A. Since A has rank 2, Col(A) is a subspace of R^3 with dimension 2. So$(Col(A))^{\perp}=Nul(A^T)$ has dimension 3 - 2 = 1. Thus $\{(3,-1,1)\}$ is a basis for $Nul(A^T)$. (This won't be on the exam).

Nul(A): x_1 and x_3 are basic variables, x_2, x_4, x_5 are free variables. Choose one free variable to be 1, and set the other free variables to 0. Then find the values of the basic variables that solve R**X = 0.** This produces 3 independent vectors in Nul(R) = Nul(A):

$s_1=\begin{bmatrix}-3\\ 1\\ 0\\ 0\\ 0\end{bmatrix}, s_2=\begin{bmatrix}-6\\ 0\\ 1\\ 1\\ 0\end{bmatrix}, s_3=\begin{bmatrix}-10\\ 0\\ 2\\ 0\\ 1\end{bmatrix}$. Then $\{s_1,s_2,s_3\}$ is a basis for Nul(A).

Next, the pivot columns of A form a basis for Col(A):

$u_1=\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$, $u_2=\begin{bmatrix}3\\ 6\\ -3\end{bmatrix}$. Check that these are orthogonal to $\begin{bmatrix}3\\ -1\\ 1\end{bmatrix}$.

The non zero rows of R form a basis for Row(A).

$$r_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 6 \\ 10 \end{bmatrix}, r_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}$$

5. Find $\begin{bmatrix} x \\ y \end{bmatrix}$ to minimize $\left\| \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} - \mathbf{b} \right\|^2$, where $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. 20 pts

This won't be on the exam, but the answer is:

$x = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \frac{1}{1^2+2^2} = \frac{7}{5}, y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \frac{1}{1^2+2^2} = \frac{10}{5} = 2$. We want $A\begin{bmatrix} x \\ y \end{bmatrix}$ to be the orthogonal projection of **b** onto Col(A). Because the columns of this A are orthogonal to each other, this projection is the sum of the projections of **b** onto each column.

6. a. Is it possible for a nonhomogeneous system of four equations in six unknowns to have a unique solution for some right-hand side? Explain. 6 pts

Answer: No. The maximum rank of the coefficient matrix A is 4, and there are 6 unknowns, so the dimension of Nul(A) is at least 2, so no solution is unique for any **b**.

b. Is it possible for such a system to have a solution for every right-hand side? Explain. 6 pts

Answer: Yes. A could have rank 4, so that Col(A) = $\mathbf{R}^4$.

7. Consider this symmetric stochastic matrix:

$$A = \begin{bmatrix} .8 & .2 \\ .2 & .8 \end{bmatrix}$$

a. What are the eigenvalues and eigenvectors of A? 12 pts

Solution: A stochastic matrix always has one eigenvalue that is 1. The other eigenvalue can be found by using the efact that the sum of all eigenvalues (including multiplicities) equals the trace of the matrix. So

$\lambda_1 + \lambda_2 = .8 + .8 = 1.6.$ Since $\lambda_1 = 1,\ \lambda_2 = 0.6.$ An eigenvector for $\lambda_1 = 1$ is in the null space of $A - I = \begin{bmatrix} -.2 & .2 \\ .2 & -.2 \end{bmatrix}$. One solution is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We should have expected this because $[1 \quad 1]A = [1 \quad 1]$ for any stochastic 2x2 matrix A, and this A is stochastic and symmetric.

An eigenvector for $\lambda_2 = 0.6$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b. Find the coordinates of $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to the basis of eigenvectors found in part a. 10 pts

Solution: Since the eigenvectors are orthogonal, we can compute the coordinates with Theorem 5, p. 339:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)}{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

So the desired coordinates are $c_1 = \frac{1}{2}, c_2 = \frac{-1}{2}$.

c. Find $\lim_{k \to \infty} A^k u_0$ 12 pts

Solution: $A^k u_0 = A^k \left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{2} 1^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{-1}{2}(.6)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ as $k \to \infty$.

8. A 3x3 symmetric matrix has a null space of dimension one containing the vector (1,1,1). Find bases and dimensions of the column space, row space, and left null space.

Solution: The row space of A is the orthogonal complement of the null space; $Row(A) = (Nul(A))^{\perp}$. The dimension of the row space is the rank of A = 3 – dim(Nul(A)) = 3-1 = 2. So we need 2 nonzero vectors orthogonal to (1,1,1). For example, $\beta = \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ is a basis for Row(A). Since A is symmetric, β is also a basis for Col(A). For the same reason the left null space ($Nul(A^T)$ is the same as Nul(A) and {(1,1,1)} is a basis for the left null space.

9. Determine whether $\{v, u, w\}$ is linearly dependent or independent.

$v = (2, \ -2, \ 3)$ $u = (3 \ \ 0, \ 4)$ $w = (1, \ -4, \ 2)$ 12 pts

Solution: Construct a matrix A with u, v, and w as column vectors. Put w first to get a 1 in the first pivot position: $A = \begin{bmatrix} 1 & 3 & 2 \\ -4 & 0 & -2 \\ 2 & 4 & 3 \end{bmatrix}$. The reduced echelon form of A is:
$R = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$. A nonzero vector in Nul(R) = Nul(A) is $\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.
So u, v, and w are linearly dependent, -w -u + v = 0.

10. Define "basis". Use a complete sentence (or two). 10 pts
A subset β of a vector space V is a basis for V if β spans V and β is linearly independent.

11. Define "linearly independent". Use a complete sentence (or two) 10 pts
A subset $\beta = \{v_1, \cdots, v_p\}$ of a vector space V is linearly independent if $c_1 v_1 + \cdots + c_p v_p = 0$ implies $c_j = 0$ for $j = 1, \cdots, p$. See also section 4.3.

12. Find the projection matrix **P** that projects any vector $\mathbf{v} \in \mathbf{R}^3$ onto the line generated by $\mathbf{b} = (2, \ -1, 2)^T$. Find the eigenvalues and eigenvectors of **P**. 15 pts
Solution: For any **v** in $\mathbf{R}^3$, the projection of **v** onto the line generated by **b** is $\left(\frac{\boldsymbol{v} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\right) \boldsymbol{b} = \frac{\boldsymbol{b}\boldsymbol{b}^T}{\boldsymbol{b}^T \boldsymbol{b}} \boldsymbol{v}$. So $\boldsymbol{P} = \frac{\boldsymbol{b}\boldsymbol{b}^T}{\boldsymbol{b}^T \boldsymbol{b}} = \frac{1}{9}\left[\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\right] =$
$\frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$

Textbook supplementary Exercises:

Ch 3, p. 285: 1ℓ-p.
Ch 4, p. 262: 1, 3, 5, 7, 9, 11, 12, 13, 14, 15.
Ch 5, p. 326: 1, 3, 5, 7, 9, 12, 13, 14, 17.
Ch 6, p. 390: 1a-q, 3, 4, 5, 6, 7,