MATH 2331 FINAL EXAM NAME______________

December 12, 2011 ID #______________

NO CALCULATORS!

1. a. Find the complete solution to the system:

$$\mathbf{BX} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

12 pts

Solution: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

b. Find a basis for the column space of B. Then express each column of B as a combination of the basis. (Hint: B has rank 2) 12 pts

Solution: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. Column 2 = Column 1;

Column 5 = Column 4 = Column 1 + Column 3.

c. Find an equation $a_1b_1 + a_2b_2 + a_3b_3 = 0$ for all vectors $\mathbf{b} = (b_1, b_2, b_3)$ for which **BX** = **b** has a solution. 12 pts

Solution: $2b_2 - b_3 = 0$. Note that $\begin{bmatrix} 0 & 2 & -1 \end{bmatrix}$ is in Nul(B^T)

d. Find a basis for Nul(B) and for its orthogonal complement. 12 pts

Solution: Basis for Nul(B): $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for $Nul(B)^{\perp}$: $\left\{\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\\1\end{bmatrix}\right\}$

3. a. Let $T:\mathbf{R}^3\to\mathbf{R}^3$ be a linear transformation such that: $T(1,0,0)=(1,2,3)$, $T(0,1,0)=(4,5,6)$, $T(0,0,1)=(7,8,9)$. Find the matrix for T. *Hint: Compute T(x,y,z).* 9 pts

Solution: $A=\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}$

b. Let $T:\mathbf{R}^2\to\mathbf{R}^3$ be a linear transformation such that: $T(1,0)=(1,-2,3)$, $T(0,1)=(-4,5,6)$. Find the standard matrix for T. *Hint: Compute T(x,y).* 8 pts

Solution: $B=\begin{bmatrix}1&-4\\-2&5\\3&6\end{bmatrix}$

4. a. Find the eigenvalues and eigenvectors of each of these matrices. Identify which are invertible and/or diagonalizable.

$A=\begin{bmatrix}3&1\\0&3\end{bmatrix}$ 6 pts

Solution: $\lambda=3,\ 3.$ Eigenvector $\begin{bmatrix}1\\0\end{bmatrix}$. Invertible and not diagonalizable.

$B=\begin{bmatrix}3&-4\\4&3\end{bmatrix}$ 6 pts

Solution: $\lambda=3+4i$, eigenvector $\begin{bmatrix}i\\1\end{bmatrix}$, $\lambda=3-4i$, eigenvector $\begin{bmatrix}-i\\1\end{bmatrix}$. Invertible and (complex) diagonalizable.

$C=\begin{bmatrix}4&2\\4&2\end{bmatrix}$ 6 pts

Solution: $\lambda = 0$, eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\lambda=6$, eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Diagonalizable and not invertible.

$D = \begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix}$ 6 pts

Solution: $\lambda = 0$, eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\lambda = 6$, eigenvector $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$.
Diagonalizable and not invertible.

$E = \begin{bmatrix} 2 & 1 \\ -17 & 4 \end{bmatrix}$ 6 pts

Solution: $\lambda = 3+4i$, eigenvector $\begin{bmatrix} 1 \\ 1+4i \end{bmatrix}$, $\lambda = 3-4i$, eigenvector $\begin{bmatrix} 1 \\ 1-4i \end{bmatrix}$.
Invertible and (complex) diagonalizable.

$F = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ 6 pts

Solution: $\lambda = 3,\ 3$, eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Invertible and not diagonalizable.

b. Organize the matrices A-F in disjoint sets of similar matrices. 8 pts
Solution: A,F; B,E; C,D.

5. a. Find the matrix P_1 that projects (x,y,z) onto the line through (1,2,2). 10 pts

Solution: $\mathbf{P}_1 = \dfrac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}} = \dfrac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}$

b. Find three eigenvalues and three independent eigenvectors for P_1. 12 pts

Solution: $\lambda = 1$, eigenvector $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\lambda = 0$, eigenvectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

c. Find the matrix P_2 that projects (x,y,z) onto the plane orthogonal to (1,2,2). 15 pts

Solution: $P_2 = I - P_1$.

6. This problem uses least squares to find the curve $y = ax + bx^2$ that best fits these 4 points in the plane: $(x_1, y_1) = (-2, 2)$, $(x_2, y_2) = (-1, 1)$, $(x_3, y_3) = (1, 0)$, $(x_4, y_4) = (2, 2)$.

a. Write down 4 equations $ax_i + bx_i^2 = y_i$, i = 1, 2, 3, 4, that would be true if the line actually went through all four points. 8 pts

Solution:
$$\begin{aligned}-2a + 4b &= 2\\ -a + b &= 1\\ a + b &= 0\\ 2a + 4b &= 2\end{aligned}$$

b. Now write those four equations in the form $\mathbf{A}\begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{y}$

Solution: $$\begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

c. Now find $\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}$ that minimizes $\left\| \mathbf{A}\begin{bmatrix} a \\ b \end{bmatrix} - \mathbf{y} \right\|^2$. 14 pts

Solution: $$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} -.1 \\ .5 \end{bmatrix}.$$

7. Consider this symmetric Markov matrix:
$$A = \begin{bmatrix} .4 & .6 \\ .6 & .4 \end{bmatrix}$$

a. What are the eigenvalues and eigenvectors of A? 8 pts

Solution: $\lambda = 1$, eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\lambda = .2$, eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b. Find $\lim_{k \to \infty} \mathbf{A}^k \mathbf{u_0}$ with $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 12 pts

Solution: $$\begin{bmatrix} .5 \\ .5 \end{bmatrix}$$

8. $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1+x & 1+x \\ 1 & 1+x & y+x & y+x \\ 1 & 1+x & y+x & y+x-z \end{bmatrix}$ Find $\det(\mathbf{A})$. 12 pts

Solution: -x(y-1)z.

9. Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{bmatrix}$. 14 pts

Solution: $\mathbf{A}^{-1} = \begin{bmatrix} 3 & -2 & 1 \\ -3 & 3 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

10. a. Find the matrix P_1 that projects (x,y,z) onto the line through (2,-1,2). See Problem 5. 12 pts

b. Find three eigenvalues and three independent eigenvectors for P_1. 12 pts

c. Find the matrix P_2 that projects (x,y,z) onto the plane orthogonal to (2,-1,2). 12 pts

11. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 3 & 2 \\ 5 & 1 & 4 & 3 & 2 \end{bmatrix}$

Find $\det(\mathbf{A})$. 12 pts

Solution: Expanding det(A) by permutations,

$$\det(\mathbf{A}) = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \det\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} = -120$$