Math 3331 Homework Solutions

1. Spring hung from ceiling. When 3-kg mass attached spring stretches 21 cm. Then mass is pulled down another 5 cm and is released from rest

a. Set up IVP $\quad m = 3\text{ kg}$.

Balance of spring force vs gravity: $k \cdot (21\text{ cm}) = 3\text{ kg} \cdot 9.8\text{ m/s}^2$

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$$k = \frac{3 \cdot 9.8}{(.21)}\text{ kg/s}^2 = \frac{9.8}{.07} = 140\text{ kg/s}^2$$

Then $3\dfrac{d^2x}{dt^2} + 140x = 0 \qquad x(0) = .05 \quad x'(0) = 0$

b. Solve the IVP: $k/m = 46.\overline{6}\ldots$, $\omega = \sqrt{46.6} \approx 6.83$.

6

$$x(t) = a\cos(6.83t) + b\sin(6.83t)$$

$x(0) = .05 = a$, $x'(0) = 0 = b \cdot 6.83 \quad b = 0$

$$x(t) = (.05)\cos(6.83t)$$

2. Use the substitution $v = y'$ to write the second order equations

$$y'' + 4y' + 5\sin(y) = 0$$

as a planar system

Solution $\quad v' = y'' = -4y' - 5\sin(y) = -4v - 5\sin(y)$.

6

So $\quad y' = v$

$\qquad v' = -4v - 5\sin(y)$

is the equivalent planar system.

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3. Use Definition 1.22, p. 141, to explain why $y_1(t)=t^3$ and $y_2(t)=t^5$ are linearly independent solutions of

$$t^2y''-7ty'(t)+15y(t)=0$$

<u>Solution</u>. According to Def'n 1.22, $y_1$ and $y_2$ are linearly dependent iff $y_1=cy_2$ or $y_2=cy_1$, where $c={}^{y_1}/_{y_2}$ is constant.

Since $\frac{d}{dt}{}^{y_2}/_{y_1}=\frac{d}{dt}t^2=2t\neq 0$, $y_1$ and $y_2$ are linearly independent.

(10)

Also $t^2y_1''-7ty_1'+15y_1=t^2\cdot 3\cdot 2t^1-7t\cdot 3t^2+15t^3$

$$=t^3(6-7\cdot 3+15)=0$$

$$t^2y_2''-7ty_2'+15y_2=t^2\,5\cdot 4\cdot t^3-7t\cdot(5t^4)+15t^5$$

$$=(20-35+15)t^5=0$$

So $y_1$ and $y_2$ are linearly independent solutions of

$$t^2y''-7ty'+15y=0.$$

4. Show that $y_1(t)=t^4$ and $y_2(t)=t$ form a fundamental set of solutions for $t^2y''-4ty'+4y=0$. Solve this with initial conditions $y(1)=2$ and $y'(1)=-3$.

(10)

<u>Solution</u> $t^2y_1''-4ty_1'+4y_1=t^2\,4\cdot 3\cdot t^2-4t\cdot 4t^3+4t^4$

$$=t^4(12-16+4)=0$$

$$t^2y_2''-4ty_2'+4y_2=t^2\cdot 0-4t\cdot 1+4t=0$$

So $y_1$ and $y_2$ are solutions. Since $y_1/y_2=t^3$ is not constant, $y_1$ and $y_2$ are linearly independent — or check that $W(y_1,y_2)\neq 0$ for $t>0$

Let $y=c_1t^4+c_2t$ $\quad y(1)=\underset{=2}{c_1+c_2}$ $\quad y'(1)=4c_1+c_2=-3$

$$\begin{array}{r} c_1+c_2=2 \\ 4c_1+c_2=-3 \\ \hline 3c_1=-5 \end{array}$$

$c_1=-5/3$

$c_2=2-c_1=2+5/3=11/3$

$\underline{Y=-5/3\,t^4+\frac{11}{3}t}$

# Math 3331 Homework Solutions

5. Use a numerical solve (pplane) to solve:

$1\cdot x'' + 6x' + 9x = 0 \quad y(0) = 3, \quad v(0) = y'(0) = 2.$

Solution: See next pages. (plot $y$ vs. $t$, $v$ vs. $t$ & combined $y$ & $v$ vs. $t$)

6. repeat #5 with $\mu = 1$ kg/s.

(3 more pages)

7. Find the general solution of these differential equations:

a. $y'' + 2y' - 15y = 0$

<u>Solution</u> The characteristic equation is

$r^2 + 2r - 15 = 0 \quad (r+5)(r-3) = 0 \quad r = 3, -5$

$y(t) = c_1 e^{-5t} + c_2 e^{3t}$

b. $y'' + 2y' + y = 0 \quad r^2 + 2r + 1 = 0 \quad (r+1)^2 = 0 \quad r = -1, -1$

$y(t) = c_1 e^{-t} + c_2 t e^{-t}$

c. $y'' + 2y' + 5y = 0 \quad r^2 + 2r + 5 = 0 \quad (r+1)^2 + 4 = 0 \quad r = -1 \pm 2i$

$y = e^{-t}(c_1 \cos 2t + c_2 \sin(2t))$

8. Solve the IVP $y'' + 16y = 0 \quad y(0) = 3, \; y'(0) = 4$.

Find the amplitude, frequency and phase of the solution

Solution $y = a\cos(4t) + b\sin(4t)$

$y(0) = 3 = a \qquad y'(0) = 4 = 4b \qquad a = 3, b = 1$

$y = 3\cos 4t + \sin 4t = \sqrt{10}\left(\frac{3}{\sqrt{10}}\cos(4t) + \frac{1}{\sqrt{10}}\sin 4t\right)$

If $\cos\phi = \frac{3}{\sqrt{10}}, \; \sin\phi = 1/\sqrt{10} \quad y = \sqrt{10}(\cos(4t - \phi))$

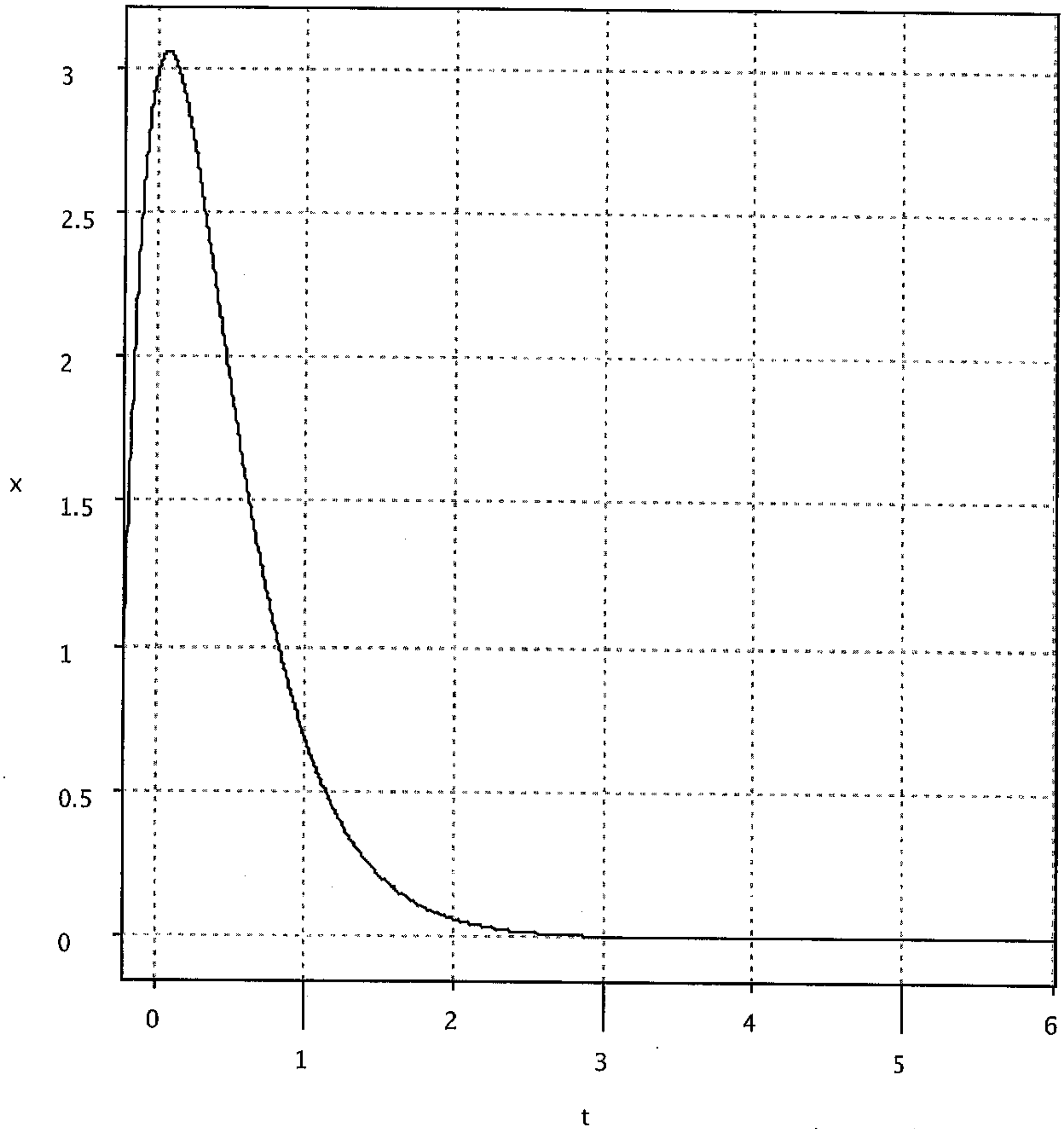
$\phi = \arctan(1/3) = .325$ radians $\quad$ Amplitude $= \sqrt{10}$

frequency $= 4$

#5

$x' = v$

$v' = -(kx + dv)/m$



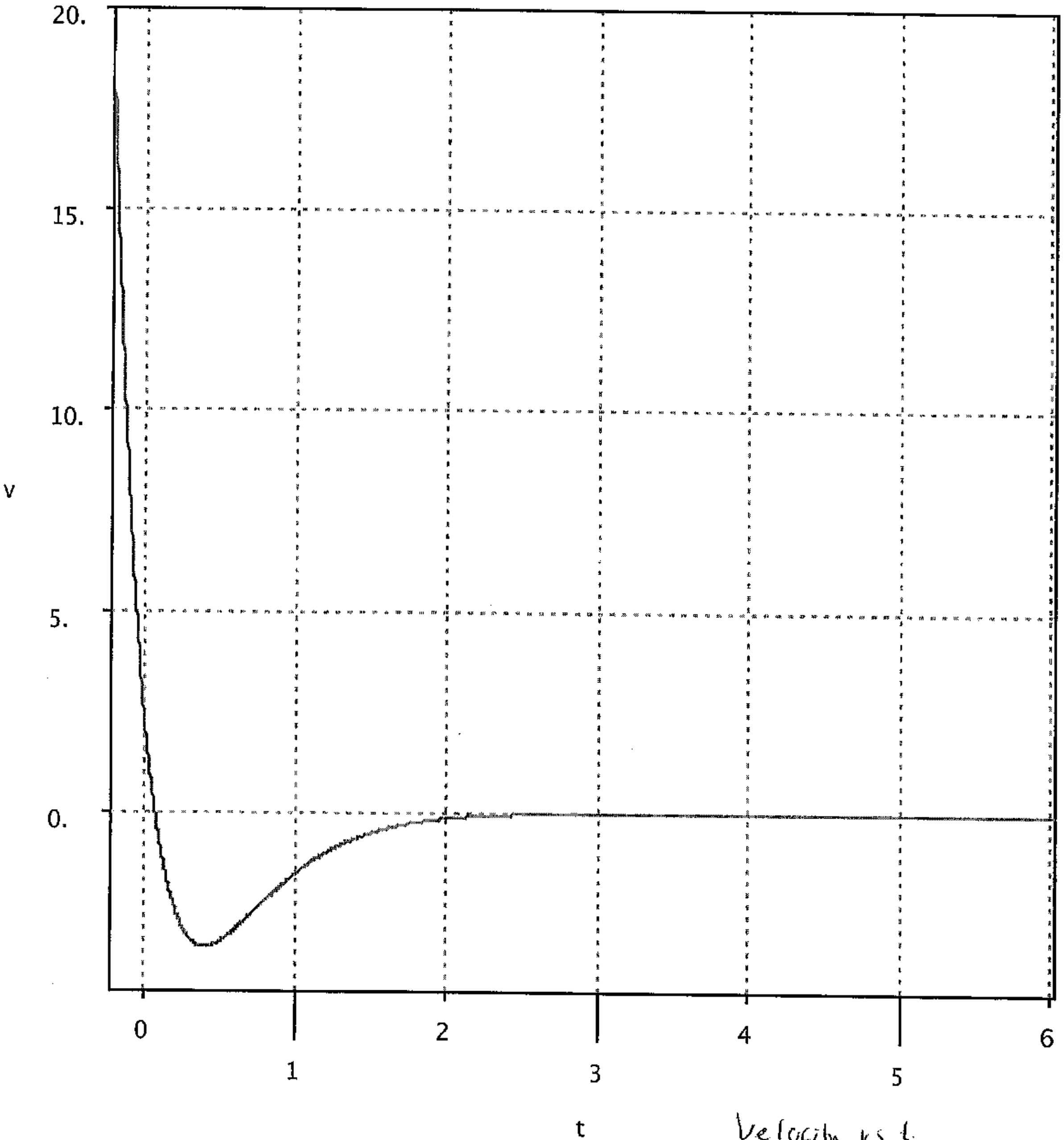

position vs t.

$\mu = 6 \quad m = 1 \quad k = 9$
$(= d)$

$x' = v$

$v' = -(kx + dv)/m$

#5



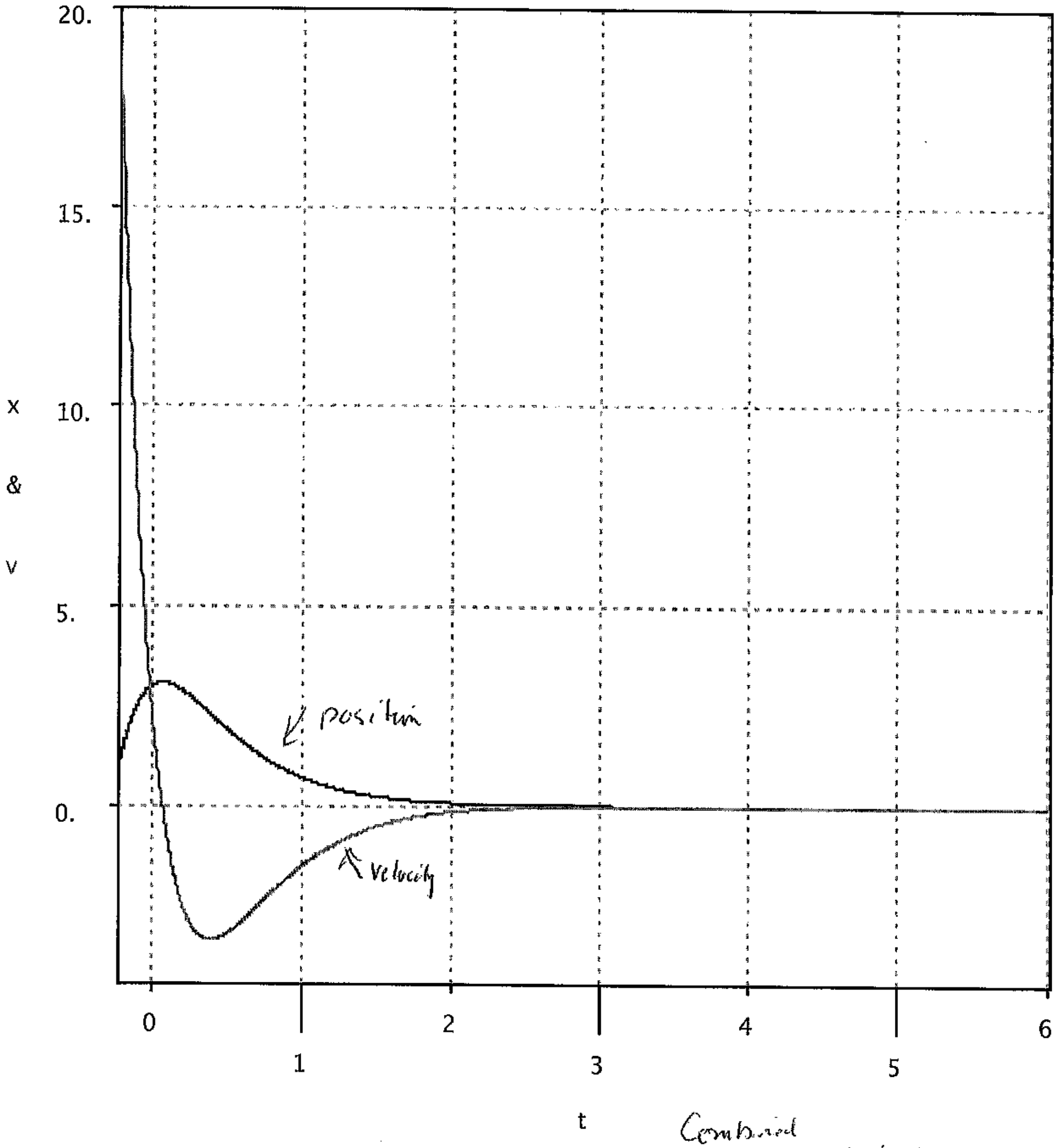

Velocity vs t.

$\mu=6$ $k=9$ $m=1$

#5

$x' = v$

$v' = -(kx + dv)/m$



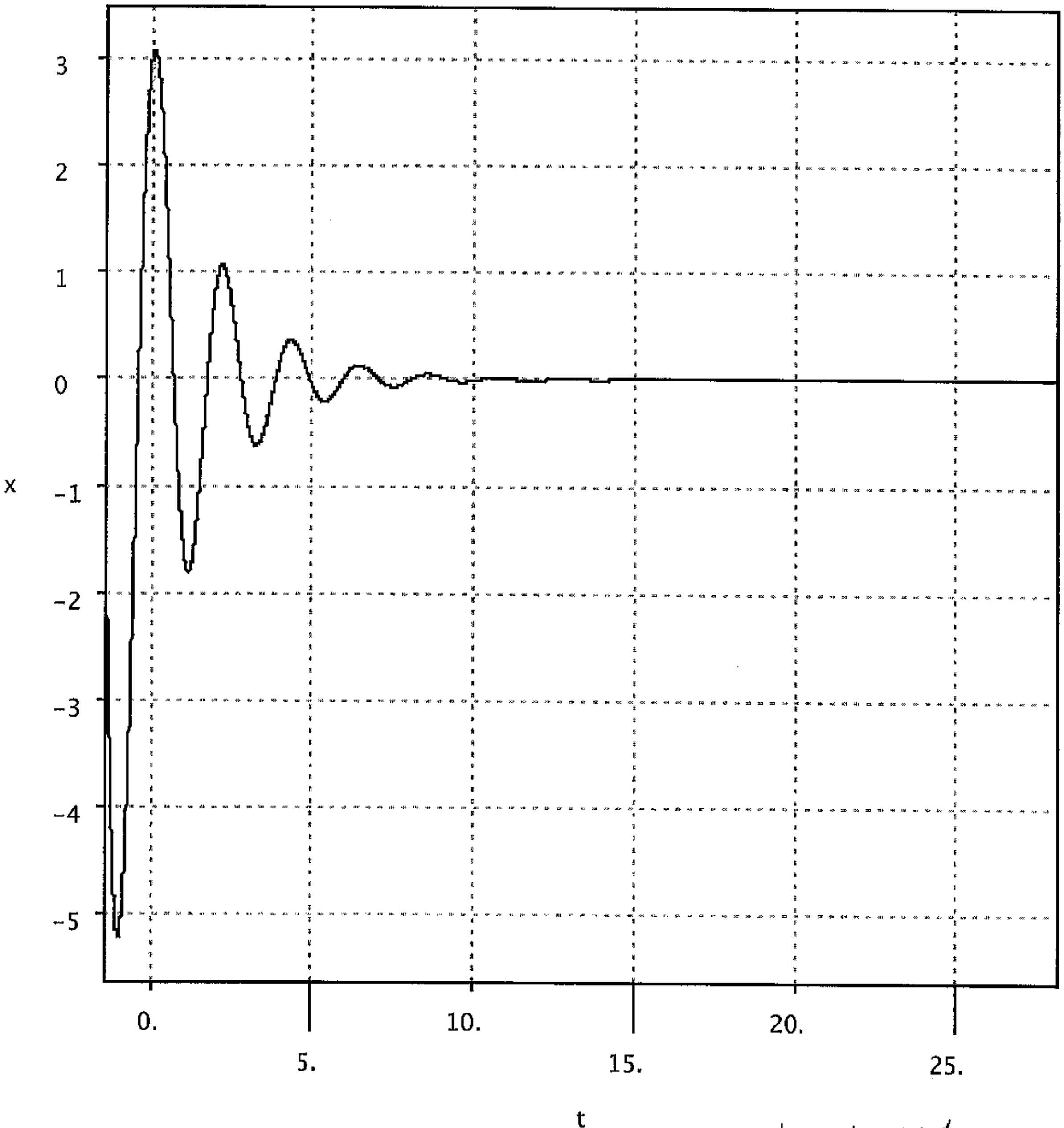

Combined
position + velocity
vs. t

$\mu=6,\ k=9\ \ m=1$

$x' = v$

#6

$v' = -(kx + dv)/m$



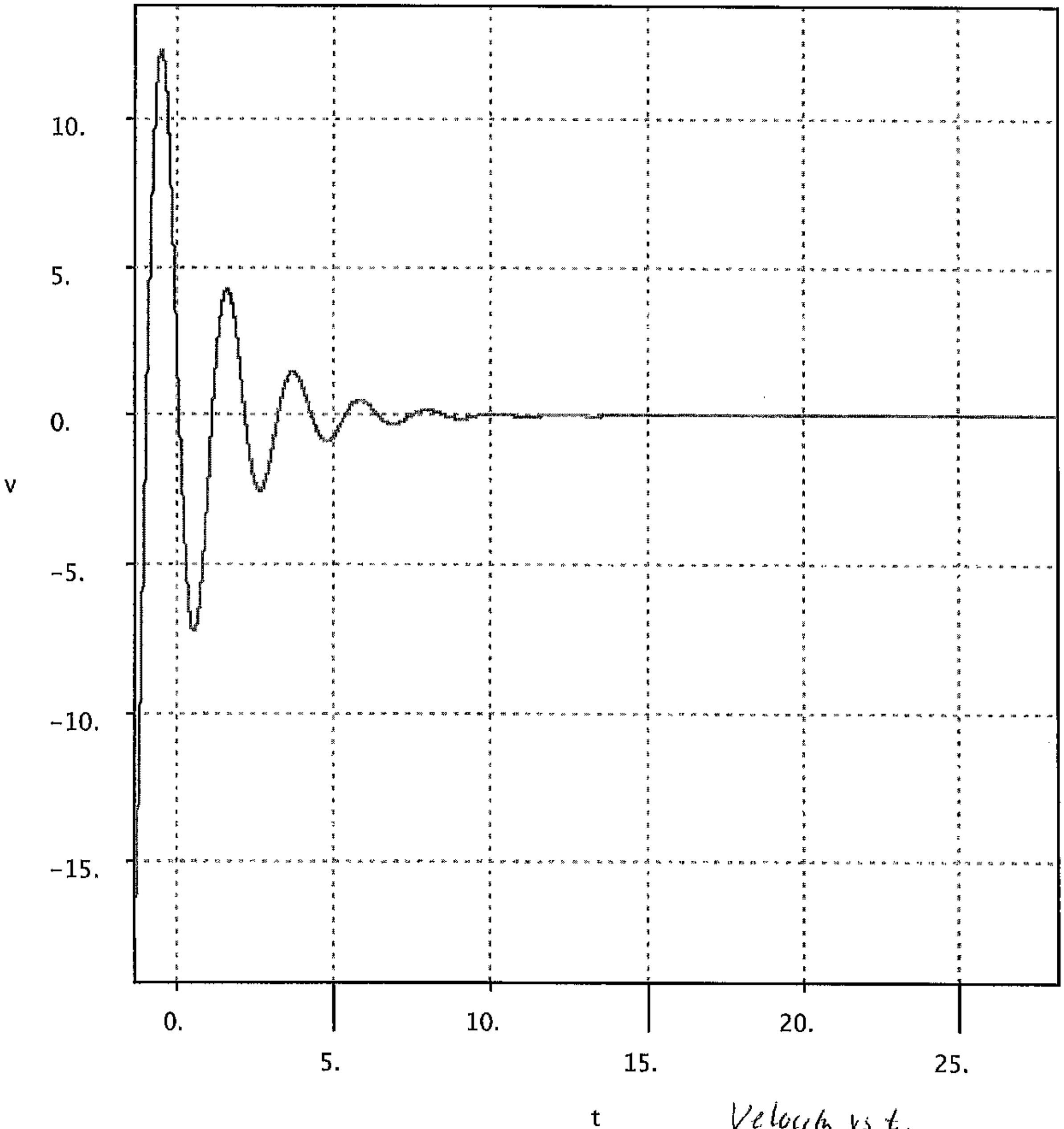

$\underline{\mu = 1}$ position vs t

$m = 1$
$k = 9$

#6

$x' = v$

$v' = -(kx + dv)/m$




Velocity vs t.

$m = 1$

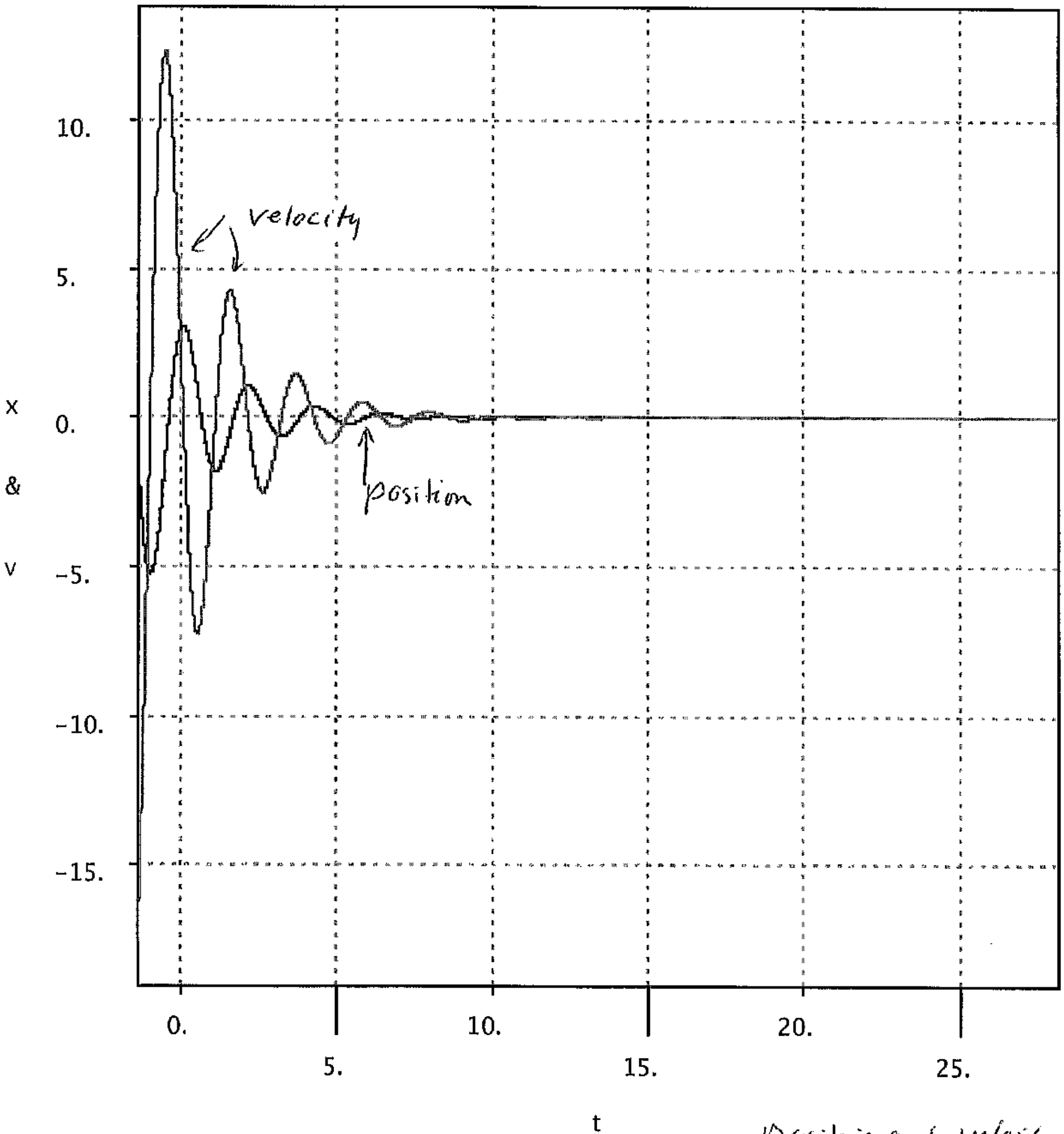
$\mu = 1$

$k = 9$

$x' = v$

$v' = -(kx + dv)/m$

#6




position and velocity vs. t