Math 3331 Homework Solutions

1. Solve the IVP $x'' + 2x' + 10x = 2\cos(2t)$
$x(0) = 0, \quad x'(0) = 0$

<u>Solution.</u>

(1) Solve homog. eqtn. $x'' + 2x' + 10x = 0$

(4)

$x = e^{rt} \rightarrow e^{rt}(r^2 + 2r + 10) = 0$

$r^2 + 2r + 10 = (r+1)^2 + 9 = 0 \quad r = -1 \pm 3i$

$x_h(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$

(8)

Try $x_p(t) = A\cos 2t + B\sin 2t$

$x_p' = -2A \sin 2t + 2B \cos 2t$

$x_p'' = -4A\cos 2t - 4B \sin 2t$

$x_p'' + 2x_p' + 10x_p = \cos 2t(-4A + 2\cdot 2B + 10)$
$+ \sin 2t(-4B - 4A + 10) = 2\cos(2t)$

So $-4A + 4B + 10 = 2 \qquad 4A - 4B = 8$
$-4A - 4B + 10 = 0 \qquad 4A + 4B = 10$
$8A = 18 \quad A = 9/4$

$4B = 10 - 4A = 10 - 9 = 1$

$B = 1/4 \qquad x_p = \frac{9}{4}\cos 2t + 1/4 \sin 2t$

(2) General Solution $x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin(3t))$
$+ 9/4 \cos 2t + 1/4 \sin 2t$

(4)

$x(0) = c_1 + 9/4 = 0 \quad c_1 = -9/4$

$x'(t) = -e^{-t}(c_1 \cos 3t + c_2 \sin 3t) - \frac{9}{2}\sin 2t + 1/2 \cos 2t$
$+ e^{-t}(3 \cdot \frac{9}{4} \sin 3t + 3c_2 \cos 3t)$

$x'(0) = -9/4 + 1/2 + 3c_2 = 0 \quad 3c_2 = 9/4 - 1/2 = 7/4$
$c_2 = 7/12$

(4)

$$x(t) = \underbrace{e^{-t}(-\tfrac{9}{4}\cos 3t + 7/12 \sin 3t)}_{\text{transcient (2)}} + \underbrace{\tfrac{9}{4}\cos 2t + 1/4 \sin 2t}_{\text{Steady state (2)}}$$

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2. Find all positive values of ω for which the forced spring with equation

$$x'' + 16x = \sin(\omega t)$$

is resonant.

Solution $\omega = 4$. If $\omega = 4$, particular solution has the

(8) form $x_p(t) = At\sin\omega t + Bt\cos(\omega t)$

$\rightarrow \quad x_p' = A\sin\omega t + B\cos(\omega t) + A\omega t\cos\omega t - B\omega t\sin\omega t$

$x_p'' = 2A\omega\cos(\omega t) - 2B\omega\sin\omega t - \omega^2 t(A\sin\omega t + B\cos\omega t)$

$x_p'' + 16x_p = 8(A\cos 4t - B\sin 4t) - 16t(A\sin 4t + B\cos 4t)$
$(\omega = 4)$
$\quad + 16(At\sin 4t + Bt\cos 4t)$

$= 8A\cos 4t - 8B\sin 4t = \sin\omega t$

$B = -1/8, A = 0$

$x_p = -t/8\cos(\omega t)$ grows unbounded

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\# 3. Find the Laplace transform of f

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ e^{2t}, & 1 \le t < \infty \end{cases}$$

Solution 1 $\mathcal{L}(f(t))(s) = \int_0^1 t\, e^{-st}dt + \int_1^\infty e^{2t} e^{-st}dt$

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$$= \frac{-t e^{-st}}{s}\Big|_0^1 + \int_0^1 \frac{e^{-st}}{s}dt + \frac{-e^{-(s-2)t}}{(s-2)}\Big|_1^\infty$$

$$= \frac{-e^{-s}}{s} - \frac{e^{-st}}{s^2}\Big|_0^1 + \frac{1}{s-2}e^{-(s-2)}$$

$$= \frac{-e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{1}{s-2}e^{-(s-2)} \qquad s > 2$$

or Solution 2 $f(t) = t\,(1 - H_1(t)) + H_1(t)e^{2t}$

$$= t + H_1(t)(e^{2t} - t)$$

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So $\mathcal{L}(f(t))(s) = \mathcal{L}(t)(s) + \mathcal{L}(H_1(t)(e^{2t} - t))$

$(e^{2t} - t) = e^{2(t-1)}e^2 - (t-1) - 1$

Then $\mathcal{L}(H_1(t)(e^{2t} - t)) = \mathcal{L}(H_1(t)(e^{2(t-1)}e^2 - (t-1) - 1))$

$= e^{-s}\mathcal{L}(e^{2t} - t - 1) = e^{-s}\left(\frac{1}{s-2} - \frac{1}{s^2} - \frac{1}{s}\right)$

So $\mathcal{L}(f(t))(s) = \frac{1}{s^2} + e^{-s}\left(\frac{1}{s-2} - \frac{1}{s^2} - \frac{1}{s}\right)$

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4. Find $f(t)$, if $\mathcal{L}(f(t))(s) = \frac{3}{s^2+2s+5}$

Solution $F(s) = \frac{3}{(s+1)^2+4}$, By Table 1 p. 204

(8) $\mathcal{L}(e^{at}\sin bt)(s) = \frac{b}{(s-a)^2+b^2}$.

We have $b=2$, $a=-1$, so $F(s) = (3/2)\frac{2}{(s+1)^2+2^2}$

So $f(t) = 3/2\, e^{-t}\sin(2t)$

5. Find $f(t)$, if $\mathcal{L}(f(t))(s) = F(s) = \frac{5s^2-11s+14}{(s-3)(s^2+4)}$.

Solution Expand $F(s)$ by partial fractions

(12) $F(s) = \frac{A}{s-3} + \frac{Bs+C}{s^2+4} = \frac{A(s^2+4)+(Bs+C)(s-3)}{(s-3)(s^2+4)}$

$\Rightarrow A(s^2+4) + (Bs+C)(s-3) = 5s^2-11s+14$

$s=3 \rightarrow 13A = 45-33+14 = 26 \Rightarrow A=2.$

Then $(Bs+C)(s-3) = 5s^2-11s+14-2s^2-8 = 3s^2-11s+6$

$Bs^2+Cs-3Bs-3C = 3s^2-11s+6$

$B=3$, $(C-3B) = -11$, $-3C=6$ $C=-2$.

$-3B = -11-C = -9$ $B=3$

So $F(s) = \frac{2}{s-3} + \frac{3s-2}{s^2+4} = \frac{2}{s-3} + 3\left(\frac{s}{s^2+4}\right) - \frac{2}{s^2+4}$

So $f(t) = 2e^{3t} + 3\cos 2t - 2\sin 2t$.