Math 3331 Homework Solutions

1. Verify that $\{\vec{y}_1(t), \vec{y}_2(t)\}$ is a linearly independent set of solutions to the given system. Then find the solution $\vec{y}(t)$ to this system with $\vec{y}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\vec{Y}_1(t) = \begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix}, \quad \vec{Y}_2(t) = \begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix}, \quad \vec{y}\,'(t) = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix} \vec{y}(t)$$

<u>Solution</u> $Y_1'(t) = \begin{pmatrix} -6e^{3t} \\ 3e^{3t} \end{pmatrix}$; $A\vec{Y}_1 = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -2e^{3t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} -6e^{3t} \\ 3e^{3t} \end{pmatrix} = \vec{Y}_1'(t)$

$Y_2'(t) = \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix}$; $A\vec{Y}_2 = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} -2e^{2t} \\ 2e^{2t} \end{pmatrix} = Y_2'(t)$

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So $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions.

Since their values at $t=0$ are $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

and are linearly independent, the set of solutions $\{\vec{y}_1(t), \vec{y}_2(t)\}$ is linearly independent

The solution that satisfies $Y(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is $c_1 Y_1(t) + c_2 Y_2(t)$,

where $c_1 Y_1(0) + c_2 Y_2(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, or

$$\begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \to \text{augmented matrix}$$

$$\begin{pmatrix} -2 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ -2 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

$c_1 = -1, \quad c_2 = 3$

$$Y(t) = -y_1(t) + 3y_2(t) = \begin{pmatrix} 2e^{3t} - 3e^{2t} \\ -e^{3t} + 3e^{2t} \end{pmatrix}.$$

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2. $A = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

2 a. $A - 3I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2 b. $(A-3I)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2 c. $(A-3I)^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

2 d. $(A-3I)^4 = \vec{0}_{4\times 4}.$

4 e. $e^{At} = e^{3It} e^{(A-3I)t}$

$$= e^{3t} I \left(I + (A-3I)t + (A-3I)^2 \tfrac{t^2}{2} + (A-3I)^3 \frac{t^3}{3!} + 0 \right)$$

$$= e^{3t} \begin{pmatrix} 1 & t & t^2/2 & t^3/3 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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3. Find the general solution of:

a. $x'(t) = \begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix} x$ and find the solution with $x(0) = \begin{bmatrix}3\\0\end{bmatrix}$

<u>Solution</u> $A = \begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix}$ has characteristic polynomial

$\lambda^2 - 5\lambda + 4$. $\lambda^2 - 5\lambda + 4 = 0$ if $\lambda = 4, 1$

$\lambda = 1$: $A - I = \begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix}$. A basis for $N(A-I)$ is $\left\{\begin{bmatrix}1\\-2\end{bmatrix}\right\}$

$\begin{bmatrix}1\\-2\end{bmatrix}e^t$ is one solution

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$\lambda = 4$: $A - 4I = \begin{bmatrix}-1 & 1\\ 2 & -2\end{bmatrix}$. basis for $N(A-4I)$ is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

Solution $\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$

General Solution $X(t) = c_1\begin{bmatrix}1\\-2\end{bmatrix}e^t + c_2\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$

$X(0) = \begin{bmatrix}1 & 1\\ -2 & 1\end{bmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix}$ $\begin{bmatrix}1 & 1 & 3\\ -2 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 1 & 3\\ 0 & 3 & 3\end{bmatrix}$

$\to \begin{bmatrix}1 & 0 & 2\\ 0 & 3 & 3\end{bmatrix} \to c_1 = 2,\ c_2 = 1$

$X(t) = 2\begin{bmatrix}1\\-2\end{bmatrix}e^t + 1\cdot\begin{bmatrix}1\\1\end{bmatrix}e^{4t}$

b. $X'(t) = \begin{pmatrix}5 & -1\\ 4 & 1\end{pmatrix}X$, initial condition $X(0) = \begin{bmatrix}5\\7\end{bmatrix}$

<u>Solution</u> $A = \begin{pmatrix}5 & -1\\ 4 & 1\end{pmatrix}$ has characteristic equation

$\lambda^2 - 6\lambda + 9 = 0$ $(\lambda - 3)^2 = 0$: $\lambda = 3, 3$

12 $A - 3I = \begin{pmatrix}2 & -1\\ 4 & -2\end{pmatrix}$. $v = \begin{bmatrix}1\\2\end{bmatrix}$ is a basis for $N(A-3I)$, $X_1(t) = \begin{pmatrix}1\\2\end{pmatrix}e^{3t}$

Let $w = \begin{pmatrix}1\\0\end{pmatrix}$, $(A-3I)w = \begin{pmatrix}2\\4\end{pmatrix}$; $X_2(t) = we^{3t} + (A-3I)w\,te^{3t}$

$X_2(t) = \begin{pmatrix}1\\0\end{pmatrix}e^{3t} + \begin{pmatrix}2\\4\end{pmatrix}te^{3t}$

General solution $X(t) = c_1\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + c_2\left(\begin{pmatrix}1\\0\end{pmatrix}e^{3t} + \begin{pmatrix}2\\4\end{pmatrix}te^{3t}\right)$

$X(0) = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}5\\7\end{bmatrix}$ $2c_1 = 7$ $c_1 = 3.5$, $c_2 = 5 - c_1 = 1.5$

$X(t) = 3.5\begin{pmatrix}1\\2\end{pmatrix}e^{3t} + 1.5e^{3t}\left(\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}2\\4\end{pmatrix}\right)$

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3c $X'(t) = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix} X$, initial value $X(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

Solution $A = \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$; Characteristic equation $\lambda^2 - 6\lambda + 25 = 0$

$(\lambda - 3)^2 + 16 = 0 \rightarrow \lambda = 3 \pm 4i$

$\lambda = 3+4i$: $A - (3+4i)I = \begin{pmatrix} -4i & 4 \\ -4 & -4i \end{pmatrix} \sim \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$

$V = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector for $\lambda = 3+4i$

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$X(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{3t} e^{4it}$ is a complex solution

$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{3t} (\cos 4t + i \sin 4t)$

$= e^{3t}\left(\begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix} + i \begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix} \right)$

$Y_1(t) = e^{3t}\begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix}$ and $Y_2(t) = e^{3t}\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}$

are 2 independent real solutions.

General solution $X(t) = c_1 e^{3t}\begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}$

$X(0) = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow c_1 = 2,\ c_2 = 5$

$X(t) = 2e^{3t}\begin{pmatrix} \cos 4t \\ -\sin 4t \end{pmatrix} + 5e^{3t}\begin{pmatrix} \sin 4t \\ \cos 4t \end{pmatrix}$

d. $X'(t) = \begin{pmatrix} 3 & -5 \\ 5 & -3 \end{pmatrix} X$, $X(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

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Solution $A = \begin{pmatrix} 3 & -5 \\ 5 & -3 \end{pmatrix}$ has char. eqn $\lambda^2 - 9 + 25 = \lambda^2 + 16 = 0$ $\lambda = \pm 4i$

$\lambda = 4i$, $A - 4i = \begin{pmatrix} 3-4i & -5 \\ 5 & -3-4i \end{pmatrix} \sim \begin{pmatrix} 5 & -3-4i \\ 0 & 0 \end{pmatrix}$

$5x_1 + (-3-4i)x_2 = 0 \Rightarrow x_1 = \frac{3+4i}{5} x_2$ $\begin{pmatrix} 3+4i \\ 5 \end{pmatrix}$ eigenvector

$X(t) = e^{4it}\begin{pmatrix} 3+4i \\ 5 \end{pmatrix} = (\cos 4t + i \sin 4t)\begin{pmatrix} 3+4i \\ 5 \end{pmatrix}$

$= \begin{pmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{pmatrix} + i \begin{pmatrix} 4\cos 4t + 3\sin 4t \\ 5\sin 4t \end{pmatrix}$

$Y_1(t) = \begin{pmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{pmatrix}$, $Y_2(t) = \begin{pmatrix} 4\cos(4t) + 3\sin 4t \\ 5\sin 4t \end{pmatrix}$

$X(t) = c_1 Y_1(t) + c_2 Y_2(t)$

$X(0) = c_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$

$c_1 = 4/5$ $c_2 = -\frac{12}{5} \cdot \frac{1}{4} = -3/5$

$X(t) = \frac{4}{5}\begin{pmatrix} 3\cos 4t - 4\sin 4t \\ 5\sin 4t \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 4\cos 4t + 3\sin 4t \\ 5\sin 4t \end{pmatrix}$