MATH 3331 HOMEWORK

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(1) The motion of a spring-mass system is modeled by the equation:

$$x''(t) + \mu x'(t) + 25x(t) = 0.$$

Find all positive values of μ for which this system is

(a) underdamped: $0 < \mu < 5$

(b) critically damped: $\mu = 5$

(c) overdamped: $\mu > 5$

(d) What is the general solution for the critically damped case?

$$x(t) = c_1 e^{-5t} + c_2 t e^{-5t}.$$

(2) Find a particular solution:

(a) $y'' + 5y' + 4y = 3e^{2t} + 5e^{4t}$

Solution: Try $y = Ae^{2t} + Be^{4t}$. Substitute in the differential equation to get:

$$Ae^{2t}(4 + 10 + 4) + Be^{4t}(16 + 20 + 4) = 3e^{2t} + 5e^{4t}.$$

Then $A = \frac{3}{18} = \frac{1}{6}$ and $B = \frac{5}{40} = \frac{1}{8}$, so

$$y_p(t) = \frac{1}{6}e^{2t} + \frac{1}{8}e^{4t}.$$

(b) $y'' + 9y = t^2$

Solution: Try $y = At^2 + Bt + C/$ Substitute in the differential equation to get:

$$A\left(2 + 9t^2\right) + 9Bt + 9C = t^2.$$

Equating like terms, we get $9At^2 = t^2$, $9Bt = 0$, $2A + 9C = 0$, so $A = \frac{1}{9}$, $B = 0$, $C = -\frac{2}{9}A = -\frac{2}{81}$. Thus

$$y_p(t) = \frac{1}{9}t^2 - \frac{2}{81}.$$

(c) $y'' + 4y = \tan(2t)$

Solution: Use "variation of parameters." The general solution to $y'' + 4y = 0$ is $y = c_1\cos(2t) + c_2\sin(2t)$. Try $y_p(t) = c_1(t)\cos(2t) + c_2(t)\sin(2t)$. Substitute in

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the differential equation $y'' + 4y = \sec(2t)$, assume $c_1'(t)\cos(2t) + c_2'(t)\sin(2t) = 0$ and find that $c_1'(t)\,(-2\sin(2t)) + c_2'(t)2\cos(2t) = \sec(2t)$. Then

$$\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \sec(2t) \end{pmatrix}.$$

Divide the second component of this equation by 2 to obtain:

$$\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \sec(2t)/2 \end{pmatrix}.$$

The matrix $\begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix}$ is a rotation matrix with inverse $\begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}$. So

$$\begin{pmatrix} c_1'(t) \\ c_2'(t) \end{pmatrix} = \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 0 \\ \sec(2t)/2 \end{pmatrix} = \begin{pmatrix} -\tan(2t)/2 \\ 1/2 \end{pmatrix}$$

Then

$$c_1(t) = \int -\tan(2t)/2 \; dt = \frac{1}{4}\ln|\cos(2t)|$$
$$c_2(t) = \int 1/2 \; dt = \frac{1}{2}t,$$

so

$$y_p(t) = \frac{1}{4}\ln|\cos(2t)|\cos(2t) + \frac{1}{2}t\sin(2t).$$