Find the derivative of each of the following functions:

(1) $f(x,y,z) = x^y;\quad Df(x,y,z) = (yx^{y-1},\ x^y\ln(x),\ 0)$

(2) $f(x) = x^x;\quad f'(x) = x^x(\ln(x)+1).$

(3) $f(x,y,z) = \sin(x\sin(y));$
$Df(x,y,z) = \cos(x\sin(y)) * (\sin(y),\ x\cos(y),\ 0).$

(4) $f(x,y,z) = \sin\big(x\sin\left(y\sin(z)\right)\big)$
$Df(x,y,z) = \cos\big(x\sin(y\sin(z))\big)*\big(\sin(y\sin(z)),\ x\cos(y\sin(z))*\sin(z),\ x\cos(y\sin(z))* y\cos(z)\big).$

(5) $f(x,y,z) = x^{(y^z)};\ f(x,y,x) = \exp\big(\ln(x)\exp\left(z\ln(y)\right)\big).$
$Df(x,y,z) = x^{(y^z)}\left(\frac{y^z}{x},\ \ln(x)y^z * \frac{z}{y},\ \ln(x)\ln(y)y^z\right)$

(6) $f(x,y,z) = (x^y)^z;\ f(x,y,x) = \exp\left(zy\ln(x)\right)$
$Df(x,y,z) = (x^y)^z * \big(\frac{zy}{x},\ z\ln(x),\ y\ln(x)\big).$

(7) $f(x,y,z) = x^{(y+z)};\ f(x,y,z) = \exp\left((y+z)\ln(x)\right).$
$Df(x,y,z) = x^{(y+z)} * \big(\frac{y+z}{x},\ \ln(x),\ \ln(x)\big).$

(8) $f(x,y,z) = (x+y)^z;\ f(x,y,z) = \exp\left(z\ln(x+y)\right).$
$Df(x,y,z) = (x+y)^z * \left(\frac{z}{x+y},\ \frac{z}{x+y}, \ln(x+y)\right).$

(9) $f(x,y) = \sin(xy);\quad Df(x,y) = \cos(xy)\,(y,x).$

Use the Fundamental Theorem of Calculus where needed to find the derivative of:

(10) $f(x,y) = \int_a^{x+y} g(t)\ dt$ (g continuous at $x+y$):
$Df(x,y) = g(x+y) * (1,1).$

(11) $f(x,y) = \int_a^{xy} g(t)\ dt$ (g continuous at xy):
$Df(x,y) = g(xy) * (y,x)$

(12) $f(x,y,z) = \int_{xy}^{\sin(x\sin(y\sin(z)))} g(t)\ dt$; g is continuous at endpoints. Let $u = y\sin(z)$

$$Df(x,y,z) = g\big(\sin(x\sin(u))\big) * \big(\cos(x\sin(u))\big) * \big(\sin(u),\ x\cos(u)*\sin(z),\ x\cos(u)*y\cos(z)\big) - g(xy)*(y,x,0)\,.$$

(13) A function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is *bilinear* if for all $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ and all $\mathbf{w},\ \mathbf{z} \in \mathbb{R}^m$, and all $a \in \mathbb{R}$:

- $f(\mathbf{x} + a\mathbf{y}, \mathbf{z}) = f(\mathbf{x}, \mathbf{z}) + af(\mathbf{y}, \mathbf{z})$
- $f(\mathbf{x}, \mathbf{w} + a\mathbf{z}) = f(\mathbf{x}, \mathbf{w}) + af(\mathbf{x}, \mathbf{z})$

(a) Prove that if f is bilinear, then

$$\lim_{(\mathbf{h},\mathbf{k})\to(0,0)} \frac{|f(\mathbf{h}, \mathbf{k})|}{|(\mathbf{h}, \mathbf{k})|} = 0.$$

Proof. Let $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$ be a basis for $\mathbb{R}^n$, and let $\mathbf{u}_1, \cdots, \mathbf{u}_m$ be a basis for $\mathbb{R}^m$. Then if $\mathbf{h} = \sum_{i=1}^n h_i \mathbf{e}_i$ and $\mathbf{k} = \sum_{j=1}^m k_j \mathbf{u}_j$,

$$f(\mathbf{h}, \mathbf{k}) = \sum_{i=1}^n \sum_{j=1}^m h_i k_j f(\mathbf{e}_i, \mathbf{u}_j)$$

Let $M = \max |f(\mathbf{e}_i, \mathbf{u}_j)|$. Then $|f(\mathbf{h}, \mathbf{k})| \leq M \sum_{i=1}^n \sum_{j=1}^m |h_i| |k_j|$. This equals $M \sum_{i=1}^n |h_i| \sum_{j=1}^m |k_j|$. By the Schwartz inequality this is $\leq M\sqrt{n} \|\mathbf{h}\| \sqrt{m} \|\mathbf{k}\| = C \|\mathbf{h}\| \|\mathbf{k}\|$. Then

$$\frac{|f(\mathbf{h}, \mathbf{k})|}{(\mathbf{h}, \mathbf{k})} \leq \frac{C \|\mathbf{h}\| \|\mathbf{k}\|}{\sqrt{\|\mathbf{h}\|^2 + \|\mathbf{k}\|^2}} = \frac{C}{\sqrt{\frac{1}{\|\mathbf{k}\|^2} + \frac{1}{\|\mathbf{h}\|^2}}}$$

and this tends to 0 as $\|\mathbf{h}\|^2 + \|\mathbf{k}\|^2 \to 0$. □

(b) Prove that (as a linear map $(\mathbf{h}, \mathbf{k}) \to Df(\mathbf{a}, \mathbf{b})(\mathbf{h}, \mathbf{k})$)

$$Df(\mathbf{a}, \mathbf{b})(\mathbf{h}, \mathbf{k}) = f(\mathbf{a}, \mathbf{k}) + f(\mathbf{h}, \mathbf{y}).$$

Proof.

$$f(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}) - f(\mathbf{a}, \mathbf{b}) = f(\mathbf{a}, \mathbf{k}) + f(\mathbf{h}, \mathbf{b}) + f(\mathbf{h}, \mathbf{k}).$$

Let $L(\mathbf{h}, \mathbf{k}) = f(\mathbf{a}, \mathbf{k}) + f(\mathbf{h}, \mathbf{b})$. Then L is a linear function of $\mathbf{h}$ and $\mathbf{k}$, and $f(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}) - f(\mathbf{a}, bfb) - L(\mathbf{h}, \mathbf{k}) = f(\mathbf{h}, \mathbf{k})$. By part a, $f(\mathbf{h}, \mathbf{k}) = \epsilon(\mathbf{h}, \mathbf{k}) \|(\mathbf{h}, \mathbf{k})\|$. Thus $L(\mathbf{h}, \mathbf{k}) = Df(\mathbf{a}, \mathbf{b})(\mathbf{h}, \mathbf{k})$. □

This brings up a question: What is the gradient $\nabla f(\mathbf{a}, \mathbf{b})$? By definition, it is a vector $\mathbf{v} \in \mathbb{R}^{n+m}$ such that $Df(\mathbf{a}, \mathbf{b})(\mathbf{h}, \mathbf{k}) = \mathbf{v} \cdot (\mathbf{h}, \mathbf{k})$. That is, $f(\mathbf{a}, \mathbf{k}) + f(\mathbf{h}, \mathbf{b}) = \mathbf{v} \cdot (\mathbf{h}, \mathbf{k})$. If $\mathbf{a} = \sum_{i=1}^n a_i \mathbf{e}_i$ then $f(\mathbf{a}, \mathbf{k}) = \sum_{i=1}^n \sum_{j=1}^m a_i k_j f(\mathbf{e}_i, \mathbf{u}_k)$. Let $f_{ij} = f(\mathbf{e}_i, \mathbf{u}_k)$. Then $f(\mathbf{a}, \mathbf{k}) = \sum_{i=1}^n \sum_{k=1}^m a_i f_{ij} h_k = \mathbf{a}^T \mathbf{F} \cdot \mathbf{h}$, where $\mathbf{F}$

is the $n \times m$ matrix with entries f_{ij}. Similarly $f(\mathbf{h}, \mathbf{b}) = \mathbf{h} \cdot F\mathbf{b}$. If we put $\mathbf{Fb}$ in the first n positions of a vector in $\mathbb{R}^{n+m}$, and we put $\mathbf{a}^T\mathbf{F}$ in the last m positions, we have that

$$f(\mathbf{a}, \mathbf{k}) + f(\mathbf{h}, \mathbf{b}) = \left(\mathbf{Fb}, \mathbf{a}^T\mathbf{F}\right) \cdot (\mathbf{h}, \mathbf{k}) .$$

So $\nabla f(\mathbf{a}, \mathbf{b}) = \left(\mathbf{Fb}, \mathbf{a}^T\mathbf{F}\right)$.