Advanced Calculus

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L'Hôpital's Rule II

Theorem

L'Hôpital's Rule is valid if $|\lim_{x\to a} f(x)| = |\lim_{x\to a} g(x)| = \infty$.

Proof worked on blackboard.

Corollary

For any $a > 0$ *we have*

$$\lim_{x\to\infty} \frac{x^a}{e^x} = \lim_{x\to\infty} \frac{\log x}{x^a} = \lim_{x\to 0} \frac{\log x}{x^{-a}} = 0.$$

Proof: $\log\left(\frac{x^a}{e^x}\right) = a\log x - x \to -\infty$, so $\frac{x^a}{e^x} \to 0$. $\frac{\log x}{x^a} \to \frac{\infty}{\infty}$, so L'Hôpital's Rule tells us to examine $\lim_{x\to\infty} \frac{1}{ax^a} = 0$. The third case is similar.

Local Extreme Values

Theorem

Let f be continuously differentiable on an open set D, and suppose that f has a local maximum (minimum) at $p \in D$. Then $\nabla f(p) = \mathbf{0}$.

Proof.

Let $g_i(h) = f(p + he_i)$. Then g_i has a local maximum or minimum at $h = 0$. But then, $\frac{\partial f}{\partial x_i}(p) = 0 \; i = 1, \cdots, n$. □

A point p such that $\nabla f(p) = \mathbf{0}$ is called a *critical point*, or *stationary point*.

Gradient and Directional Derivatives

Theorem

Let f be differentiable at p. The the largest directional derivative of f at p occurs in the direction of $\nabla f(p)$ and has value $\|\nabla f(p)\|$

Proof.

$$D_{\mathbf{u}} f(p) = \nabla f(p) \cdot \mathbf{u} \leq \|\nabla f(p)\| \, \|\mathbf{u}\| = \|\nabla f(p)\| \, ,$$

by the Schwartz Inequality. If $\mathbf{u} = \frac{\nabla f(p)}{\|\nabla f(p)\|}$,

$D_{\mathbf{u}} f(p) = \nabla f(p) \cdot \frac{\nabla f(p)}{\|\nabla f(p)\|} = \|\nabla f(p)\|$. □

Example

Example

Suppose a hill has altitude function $z = h(x, y) = 2,000 - \frac{x^2}{900} - \frac{y^2}{1600}$, $\frac{x^2}{900} + \frac{y^2}{1600} \leq 2000$. A hiker starts at $(x = 900, y = 1600)$. She wants to climb the steepest path up the hill. What should her initial direction be?

$\nabla h(x, y) = -\left(\frac{2x}{900}, \frac{2y}{1600}\right)$; $\nabla h(900, 1600) = -(2, 2)$. The norm of this vector is $2\sqrt{2}$. The hiker should start in the direction $-\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. The slope $\left(\frac{\text{rise}}{\text{run}}\right)$ of her initial ascent will be $2\sqrt{2}$ (pretty steep!).

Differentials

Definition

Suppose f is differentiable at $\mathbf{a}$, so that

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{h} + \epsilon(\mathbf{h}) \left\|\mathbf{h}\right\| .$$

Then the linear map $\mathbf{h} \to \nabla f(\mathbf{a}) \cdot \mathbf{h}$ is called the *differential* of f at $\mathbf{a}$ and is denoted by $df_{\mathbf{a}}(\mathbf{h})$. This is the same as $Df(\mathbf{a})(\mathbf{h})$

Example

Let $f(x, y) = x$. Then f is differentiable on $\mathbb{R}^2$, $\nabla f = (1, 0)$, and $df_{(3,4)}(h, k) = h$. Actually $df = dx$.

Example

Example

Let $g(x,y) = x^3y$. Then g is differentiable on $\mathbb{R}^2$,
$\nabla g = \left(3x^2y, x^3\right)$,
$dg_{(x,y)}(h,k) = 3x^2yh + x^3k = 3x^2ydx(h,k) + x^3dy(h,k)$. So we may write:

$$dg_{(x,y)} = 3x^2ydx + x^3dy.$$

This illustrates how differentials are frequently expressed in terms of differentials of coordinates.

$$df\left(x_1, \cdots, x_n\right) = \frac{\partial f}{\partial x_1}\left(x_1, \cdots, x_n\right)dx_1 + \cdots + \frac{\partial f}{\partial x_n}\left(x_1, \cdots, x_n\right)dx_n.$$

More examples

One can look at an equation

$$dz = df(x, y) = \frac{\partial f}{\partial x}(x, y)dx + \frac{\partial f}{\partial y}(x, y)dy$$

as the equation of the tangent plane to the graph of f, with coordinates (dx, dy, dz) centered at $(x, y, f(x, y))$. This equation is equivalent to

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Rules about partial derivatives of sums, products, and quotients translate into similar rules for differentials:

$$d(f+g) = df+dg, \qquad d(fg) = fdg+gdf, \qquad d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}.$$

Differentials are handy for quick approximations to small changes in a function.

Example

A box is made with dimensions 1 foot x 2 feet x 3 feet. The measurements are accurate within 0.1 feet. How accurate is the volume calculated from these measurements?

Example

Example

The calculated volume is 6 cubic feet, $V(x, y, z) = xyz$, $dV_{(x,y,z)} = yzdx + xzdy + xydz$ or $dV = 6dx + 3dy + 2dz$. If $dx = dy = dz = 0.1$, $dV = 1.1$ feet.

The actual change in V from $V(1, 2, 3)$ to $V(1.1, 2.1, 3.1)$ is $7.161 - 6 = 1.161$. Our approximation is pretty close.

Theorem (Chain Rule version I)

Suppose $g : (a-\epsilon, a+\epsilon) \to \mathbb{R}^n$, $g(a) = \mathbf{b}$, *and* g *is differentiable at* a. *Suppose* f *maps a neighborhood of* $\mathbf{b}$ *to* $\mathbb{R}$ *and* f *is continuously differentiable at* $\mathbf{b}$. *Let* $\phi(t) = f(g(t))$. *Then* ϕ *is differentiable at* a *and* $\phi'(t) = \nabla f(\mathbf{b}) \cdot g'(a)$.

Remark

One might write $\phi'(a) = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt}(a) + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}(a)$, or if $w = f(\mathbf{x})$, $\frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}$

Proof

Proof: For simplicity, let $n = 2$. Fix $\mathbf{a} = (x, y) \in U$, and let $\epsilon > 0$. For $\mathbf{h} = (h_1, h_2)$ near $\mathbf{0}$,

$$f(\mathbf{a}+\mathbf{h})-f(\mathbf{x}) = f(x+h_1, y+h_2)-f(x, y+h_2)+f(x, y+h_2)-f(x, y).$$

By the MVT there are numbers c_1 between 0 and h_1, and c_2 between 0 and h_2 such that:

$$f(x + h_1, y + h_2) - f(x, y + h_2) = \frac{\partial f}{\partial x}(x + c_1, y + h_2)\, h_1$$
$$f(x, y + h_2) - f(x, y) = \frac{\partial f}{\partial y}(x, y + c_2)\, h_2.$$

Proof continued

Then

$$f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \nabla f \cdot \mathbf{h} = \left(\frac{\partial f}{\partial x}(x + c_1, y + h_2) - \frac{\partial f}{\partial x}(\mathbf{a}) \right) h_1 + \left(\frac{\partial f}{\partial x}(x, y + c_2) - \frac{\partial f}{\partial y}(\mathbf{a}) \right) h_2.$$

Since $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on U, and c_i is between 0 and h_i, the factors of h_1 and h_2 tend to 0 as $\mathbf{h} \to 0$. So the right hand side is $\epsilon(\mathbf{h}) \|\mathbf{h}\|$. □

Directional Derivatives

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Definition

Let D be an open subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$. We say that f has a *directional derivative* at $\mathbf{x} \in D$ in the direction $\mathbf{u}$, if

$$\lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \text{ exists.}$$

We denote this limit by $(D_{\mathbf{u}} f)(\mathbf{x})$.

The partial derivatives of f are examples of directional derivatives (when the coordinate system is orthonormal). Note that the existence or value of a directional derivative does not depend on a choice of coordinates.

Directional Derivatives

Theorem

If f is differentiable at $\mathbf{x}$, then for each unit vector $\mathbf{u}$, $(D_{\mathbf{u}}f)(\mathbf{x})$ exists and equals $\nabla f(\mathbf{x}) \cdot \mathbf{u}$.

Proof.

$$\frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} = \nabla f(\mathbf{x}) \cdot \mathbf{u} + \epsilon(h\mathbf{u}).$$

In the limit as $h \to 0$, we obtain the stated result. □

Tangent plane

Let f be differentiable at $\mathbf{x}$. When $n = 2$ and $\mathbf{x}$ fixed, the graph of $z = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h}$ is a plane in $\mathbb{R}^3$ tangent to the graph of $z = f(\mathbf{x} + \mathbf{h})$ at $(\mathbf{h} = 0, z = f(\mathbf{x}))$. If $\mathbf{u}$ is a unit vector in $\mathbb{R}^2$, the line with direction vector $(\mathbf{u}, (D_{\mathbf{u}} f)(\mathbf{x}))$ through $(\mathbf{h} = \mathbf{0}, z = f(\mathbf{x}))$ is in the tangent plane because $(D_{\mathbf{u}} f)(\mathbf{x}) = \nabla f \cdot \mathbf{u}$.

Mean Value Theorem

Theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there is at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof.

The line through $(a, f(a))$ and $(b, f(b))$ has slope $\frac{f(b)-f(a)}{b-a}$. Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a, b]$ and differentiable on (a, b). $g(b) = g(a)$, so Rolle's Theorem states that there must be $c \in (a, b)$ such that $g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. □

Applications of MVT

Definition

We say that a function f is *increasing* (respectively *strictly increasing*) on an interval I, if $f(a) \leq f(b)$ (resp. $f(a) < f(b)$) whenever $a, \ b \in I$ and $a < b$. We define *decreasing* and *strictly decreasing* functions similarly.

Theorem (Interpretation of the derivative)

Suppose f is differentiable on the open interval I.

a *If $|f'(x)| \leq C$ for all $x \in I$, then $|f(b) - f(a)| \leq C\,|b - a|$ for all $a, \ b \in I$.*

b *If $f'(x) = 0$ for all $x \in I$, then f is constant on I.*

c *If $f'(x) \geq 0$ (resp. $f'(x) > 0$, $f'(x) \leq 0$, or $f'(x) < 0$) for all $x \in I$, then f is increasing (resp. strictly increasing, decreasing, strictly decreasing) on I.*

Proof

Proof.

Let $a, \ b \in I$. Since f is differentiable on I and $[a, b] \subset I$, f is continuous on I. Then the MVT gives us a point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$. For a, $|f'(c)| \leq C$, so $|f(b) - f(a)| \leq C\,|b - a|$. Parts b, c are proved similarly. □

Remark

If all we know about f is that f is differentiable at a and $f'(a) > 0$, it does *not* follow that f is increasing in some neighborhood of a.

Generalized MVT

Theorem (Generalized MVT)

Suppose that f and g are continuous on $[a,b]$, differentiable on (a,b), and that $g'(x) \neq 0$ for all $x \in (a,b)$. Then there is $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof of Generalized MVT

Proof

- Let $h(x) = (f(b) - f(a))\,(g(x) - g(a)) - (g(b) - g(a))\,(f(x) - f(a))$.
- Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$.
- By Rolle's Theorem there is $c \in (a, b)$ such that $h'(c) = (f(b) - f(a))\,g'(c) - (g(b) - g(a))\,f'(c) = 0$.
- Since g' is never 0 on (a, b), $g'(c) \neq 0$ and $g(b) - g(a) \neq 0$ (by MVT).
- Then dividing by $g'(c)\,(g(b) - g(a))$ gives the result.

Remark

If we use f and g to parameterize a curve: $y = f(t)$, $x = g(t)$, $t \in [a, b]$, then by the chain rule, $\frac{dy}{dx} = \frac{f'(t)}{g'(t)}$. In this case the Generalized MVT says that there is a tangent line to the parameterized curve, that is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$.

Application of Generalized MVT

Theorem (L'Hôpital's Rule I)

Suppose f and g are differentiable functions on (a, b) and

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0.$$

If g' never vanishes on (a, b) and

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L,$$

then g never vanishes on (a, b) and

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L.$$