Advanced Calculus

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Multi-D Mean Value Theorem

Theorem (Mean Value Theorem)

Let U be an open set in $\mathbb{R}^n$, and suppose that the line segment $[\mathbf{a}, \mathbf{b}] \subset U$. Let f be differentiable on U. Then there is a point $\mathbf{c} \in (\mathbf{a}, \mathbf{b})$ such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a}) .$$

Proof

- Let $g(t) = (1 - t)\mathbf{a} + t\mathbf{b}$, $0 \leq t \leq 1$, so that g parameterizes $[\mathbf{a}, \mathbf{b}]$.
- Let $\phi(t) = f(g(t))$. We need to show that ϕ is differentiable on $(0, 1)$.

MVT Continued!

Let $t \in (0, 1)$.

$$\begin{aligned}
\phi(t+h) - \phi(t) &= f\left(\mathbf{a} + (t+h)(\mathbf{b}-\mathbf{a})\right) - f\left(\mathbf{a} + t(\mathbf{b}-\mathbf{a})\right) \\
&= f\left(\mathbf{a} + t(\mathbf{b}-\mathbf{a}) + h(\mathbf{b}-\mathbf{a})\right) - f\left(\mathbf{a} + t(\mathbf{b}-\mathbf{a})\right) \\
&= \nabla f\left(\mathbf{a} + t(\mathbf{b}-\mathbf{a})\right) \cdot h(\mathbf{b}-\mathbf{a}) \\
&\qquad + \epsilon\left(h(\mathbf{b}-\mathbf{a})\right) \left\| h(\mathbf{b}-\mathbf{a}) \right\| .
\end{aligned}$$

The term $\epsilon\left(h(\mathbf{b}-\mathbf{a})\right) \left\| h(\mathbf{b}-\mathbf{a}) \right\| = \epsilon\left(h(\mathbf{b}-\mathbf{a})\right) |h| \left\| \mathbf{b}-\mathbf{a} \right\|$ is clearly of the form $\epsilon(h)|h|$. So ϕ is differentiable at any $t \in (0, 1)$, and $\phi'(t) = \nabla f\left(\mathbf{a} + t(\mathbf{b}-\mathbf{a})\right) \cdot (\mathbf{b}-\mathbf{a})$.

End of MVT Proof

Now we apply the $1 - D$ MVT to ϕ: There is $c \in (0,1)$ such that:

$\phi(1) - \phi(0) = \phi'(c)$, or $f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{a} + c(\mathbf{b} - \mathbf{a})) \cdot (\mathbf{b} - \mathbf{a})$.

Thus, with $\mathbf{c} = \mathbf{a} + c(\mathbf{b} - \mathbf{a})$, the MVT is proved.

Remark

The MVT is *not* valid for vector valued functions, because a different $\mathbf{c}$ is needed for each vector component.

Theorem (Chain Rule version I)

Suppose $g : (a-\epsilon, a+\epsilon) \to \mathbb{R}^n$, $g(a) = \mathbf{b}$, and g is differentiable at a. Suppose f maps a neighborhood of $\mathbf{b}$ to $\mathbb{R}$ and f is continuously differentiable at $\mathbf{b}$. Let $\phi(t) = f(g(t))$. Then ϕ is differentiable at a and $\phi'(t) = \nabla f(\mathbf{b}) \cdot g'(a)$.

Remark

One might write $\phi'(a) = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt}(a) + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}(a)$, or if $w = f(\mathbf{x})$, $\frac{\partial w}{\partial x_1}\frac{dx_1}{dt} + \cdots + \frac{\partial w}{\partial x_n}\frac{dx_n}{dt}$

Proof

Proof:

$$\phi(a+u)-\phi(a)=f((g(a+u))-f(g(a)).$$

By the MVT there is $\mathbf{c}\in[f(g(a)),f(g(a+u))]$ such that:

$$\begin{aligned} f(g(a+u))-f(g(a)) &= \nabla f(\mathbf{c})\cdot(g(a+u)-g(a)) \\ &= \nabla f(\mathbf{c})\cdot\left(g'(a)u+\epsilon(u)u\right) \\ &= \nabla f(\mathbf{c})\cdot g'(a)u+\epsilon_2(u)u. \end{aligned}$$

Since f is C^1 at $g'(a)=\mathbf{b}$, and $\mathbf{c}\to\mathbf{b}$ as $u\to 0$,
$\nabla f(\mathbf{c})\to\nabla f(\mathbf{b})$. □